Schwarz rearrangement does not decrease the energy for the pseudo $p$-Laplacian operator

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Abstract: It is well known that the Schwarz symmetrization decrease the energy for the $p$-Laplacian operator, i.e

$$\int_{\Omega} |\nabla u|^p \, dx \geq \int_{\Omega} |\nabla u^*|^p \, dx.$$ 

where $u^*$ is the Schwarz rearranged function of $u$, for appropriate $u$ and $\Omega$. In this note, we shall prove that the Schwarz rearrangement does not decrease the energy for the pseudo $p$-Laplacian operator, that is, there exist a bounded domain $\Omega \subset \mathbb{R}^N$ and a function $u \in W^{1,p}_0(\Omega)$ such that

$$\int_{\Omega^*} \left( \sum_{i=1}^n \left| \frac{\partial u^*}{\partial x_i} \right|^p \right) \, dx \geq \int_{\Omega} \left( \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p \right) \, dx.$$ 

Key Words: Schwarz symmetrization, pseudo $p$-Laplacian operator.

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1. Introduction

The rearrangement method is defined by replacing a given function $u$ by a related function $u^*$ which has some properties like monotonicity or symmetry. The function $u^*$ can be reconstructed from its level sets

$$\Omega_c = \{ x \in \Omega \mid u(x) \geq c \}.$$ 

1.1. A CATALOGUE OF REARRANGEMENT. In the litterature we find many type of rearrangement,

1. circular and spherical symmetrization,
2. monotone decreasing rearrangement in direction $y$,
3. radial symmetrization,
4. Schwarz symmetrization,

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5. Steiner symmetrization in direction $y$.

More details of rearrangement can be found in [3,4,5,7]. In this note we are interested only by Schwarz symmetrization (the most frequently used kind symmetrization). Hence, for a Lebesgue measurable set $D \subset \mathbb{R}^n$ we define the Schwarz symmetrization $D^*$ of $D$ by

$$D^* = \begin{cases} B(0, R) & \text{if } D \neq \emptyset \\ \emptyset & \text{if } D = \emptyset \end{cases}$$

where $B(0, R)$ is a ball of $\mathbb{R}^n$ with center in the origin with same $n-\text{dim.}$ Lebesgue measure and for a Lipschitz continuous function $u$, the rearranged function $u^*$ is defined as follows

$$u^*(x) = \sup \{ c \in \mathbb{R} \mid x \in \Omega^*_c \} \text{ for } x \in \Omega^*$$

1.2. Some results for Schwarz symmetrization. One of the first powerful applications of Schwarz symmetrization was the proof of the Krahn-Faber inequality [6]: Among all fixed membranes of given area, the circular one has the lowest principal eigenvalue. This was shown by looking at

$$\lambda_1(\Omega) = \min_{u \in W^{1,2}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\|u\|^2}.$$ 

One can easily conclude, with the two following propositions, that $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$.

**Proposition 1.1** ([1,5,7]) For every continuous mapping $F : \mathbb{R}^+ \to \mathbb{R}$ and every nonnegative function $u : \Omega \to \mathbb{R}^+$, then,

$$\int_{\Omega} F(u) \, dx = \int_{\Omega^*} F(u^*) \, dx.$$

**Proposition 1.2** ([1,5,7]) For $u \neq 0 \in W^{1,p}_0(\Omega)$ and $p > 1$, we have

$$E_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx \geq \frac{1}{p} \int_{\Omega^*} |\nabla u^*|^p \, dx = E_p(u^*)$$

Then, Schwarz symmetrization decrease the potential energy $E_p(u)$ of $p$-Laplacian operator $\Delta_p u$ defined by

$$\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right).$$

The same question is posed for the so called pseudo $p$-Laplacian operator defined by

$$\Delta'_p u = \text{div} \left( \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

We show, in this paper, that the answer is negative by exhibiting an explicit function with Schwarz’s symmetrization does not decrease the energy for pseudo $p$-Laplacian operator.
2. Main result

**Theorem 2.1** The Schwarz rearrangement does not decrease the energy for the pseudo p-Laplacian operator, that is, there exist a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $u \in W_0^{1,p}(\Omega)$ such that

$$\sum_{i=1}^{n} \int_{\Omega^*} \left| \frac{\partial u^*}{\partial x_i} \right|^p \, dx \geq \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p \, dx. \quad (1)$$

where $\Omega^*$ and $u^*$ are the Schwarz rearrangement of $\Omega$ and $u$ respectively.

**Proof.** Let

$$\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq |x_1| + |x_2| \leq \sqrt{\pi} \} \quad \text{and} \quad u(x_1, x_2) = \sqrt{\pi} - (|x_1| + |x_2|),$$

$u$ is a Lipschitz continuous function that $\Omega \in W_0^{1,p}(\Omega)$. Then the Schwarz rearrangement of $\Omega$ is $\Omega^* = B(0, \sqrt{2})$.

Level sets of $u$ are given by

$$\Omega_c = \{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| < \sqrt{\pi} - c \}.$$

Then, $|\Omega_c| = \text{meas}(\Omega_c) = 2(\sqrt{\pi} - c)^2$, so

$$u^*(x_1, x_2) = \sqrt{\pi} - \sqrt{\frac{\pi}{2}}(x_1^2 + x_2^2)^{\frac{1}{2}}.$$

Now we have

$$\left| \frac{\partial u}{\partial x_1} \right| = 1 = \left| \frac{\partial u}{\partial x_2} \right|,$$

then,

$$\sum_{i=1}^{2} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p \, dx = 2\text{meas}(\Omega) = 4\pi \quad (2)$$

in the other hand,

$$\left| \frac{\partial u^*}{\partial x_1} \right| = \sqrt{\frac{\pi}{2}} \frac{|x_1|}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \quad \text{and} \quad \left| \frac{\partial u^*}{\partial x_2} \right| = \sqrt{\frac{\pi}{2}} \frac{|x_2|}{(x_1^2 + x_2^2)^{\frac{1}{2}}}$$

by passing to polar coordinates we obtain

$$\sum_{i=1}^{2} \int_{\Omega^*} \left| \frac{\partial u^*}{\partial x_i} \right|^p \, dx = \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \left( \int_{0}^{2\pi} |\cos \theta|^p \, d\theta \int_{0}^{\sqrt{2}} r \, dr + \int_{0}^{2\pi} |\sin \theta|^p \, d\theta \int_{0}^{\sqrt{2}} r \, dr \right)$$

$$= \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \left( \int_{0}^{2\pi} |\cos \theta|^p \, d\theta + \int_{0}^{2\pi} |\sin \theta|^p \, d\theta \right) \left[ \frac{1}{2} r^2 \right]_{0}^{\sqrt{2}}$$

$$= 4 \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \left( \int_{0}^{\pi} |\cos \theta|^p \, d\theta + \int_{0}^{\pi} |\sin \theta|^p \, d\theta \right)$$

$$= 8 \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \int_{0}^{\pi} |\sin \theta|^p \, d\theta \left( \int_{0}^{\pi} |\cos \theta|^p \, d\theta = \int_{0}^{\pi} |\sin \theta|^p \, d\theta \right)$$
we have used the following equality
\[
\int_0^{2\pi} |\cos \theta|^p \, d\theta = \int_0^\pi |\sin \theta|^p \, d\theta
\]
\[
= \int_0^\pi |\sin \theta|^p \, d\theta + \int_\pi^{2\pi} |\sin \theta|^p \, d\theta
\]
\[
= 2 \int_0^\pi |\sin \theta|^p \, d\theta \quad \text{(put in the second integral } \theta' = \theta - \pi)\]
\[
= 2 \left( \int_0^{\pi/2} |\sin \theta|^p \, d\theta + \int_{\pi/2}^{\pi} |\sin \theta|^p \, d\theta \right)
\]
\[
= 2 \left( \int_0^{\pi/2} |\sin \theta|^p \, d\theta + \int_0^{\pi/2} |\sin \theta|^p \, d\theta \right) \quad \text{(put in the second integral } \theta' = \pi - \theta)\]
\[
= 4 \int_0^{\pi/2} |\sin \theta|^p \, d\theta,
\]
so,
\[
\sum_{i=1}^{2} \int_{\Omega_i} \left| \frac{\partial u^*}{\partial x_i} \right|^p \, dx = 8 \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \int_0^{\pi/2} |\sin \theta|^p \, d\theta \quad \text{(3)}
\]
the integral in the right member of equation (3) is given by the well known Wallis formula \[2\] (page : 15)
\[
\int_0^{\pi/2} |\sin \theta|^p \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2} + 1\right)}.
\]
Finally,
\[
\sum_{i=1}^{2} \int_{\Omega_i} \left| \frac{\partial u^*}{\partial x_i} \right|^p \, dx = 4 \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \sqrt{\pi} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2} + 1\right)} \quad \text{(4)}
\]
the function \(\Gamma(\alpha)\) is increasing for \(\alpha \geq \frac{3}{2}\) and \(\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)\) for all \(\alpha > -1\) so,
\[
\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2} + 1\right)} \geq \frac{\Gamma\left(\frac{p}{2}\right)}{\frac{p}{2} \Gamma\left(\frac{p}{2}\right)} = \frac{2}{p},
\]
applying equation (4) to get
\[
\sum_{i=1}^{2} \int_{\Omega_i} \left| \frac{\partial u^*}{\partial x_i} \right|^p \, dx \geq \frac{8}{p} \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \sqrt{\pi}
\]
then the inequality
\[
\sum_{i=1}^{2} \int_{\Omega_i} \left| \frac{\partial u^*}{\partial x_i} \right|^p \, dx \geq \frac{2}{p} \sum_{i=1}^{2} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p \, dx
\]
is verified if
\[
\frac{8}{p} \left( \frac{\pi}{2} \right)^{\frac{p}{2}} \sqrt{\pi} \geq 4\pi
\]
The pseudo $p$-Laplacian operator

which is equivalent to

$$\left(\frac{\pi}{2}\right)^{\frac{p}{2}} - \frac{p}{2}\sqrt{\pi} \geq 0. \quad (5)$$

An elementary study of the function $(x = \frac{p}{2})$

$$f(x) = \left(\frac{\pi}{2}\right)^x - x\sqrt{\pi}$$

shows that $f$ is strictly increasing in $[\frac{1}{\ln 2}, +\infty]$. Equation (5) is then established if $f\left(\frac{p}{2}\right) \geq 0$, that is $p \geq p_c$ where $p_c$ is defined by $f\left(\frac{p_c}{2}\right) = 0$. Mean value theorem shows that $p_c \in [9, 10]$. Consequently, for all $p \geq 10$

$$\sum_{i=1}^{2} \int_{\Omega} \left| \frac{\partial u^*}{\partial x_i} \right|^p dx \geq \sum_{i=1}^{2} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p dx.$$

Conclusion: the Schwarz rearrangement does not decrease the energy for the pseudo $p$-Laplacian operator like it does for $p$-Laplacian operator.

References


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