The semi normed space defined by entire sequences

N. Subramanian, K. Chandrasekhara Rao and K. Balasubramanian

Abstract: In this paper we introduce the sequence spaces $\Gamma(p, \sigma, q, s)$, $\Lambda(p, \sigma, q, s)$ and define a semi normed space $(X, q)$, semi normed by $q$. We study some properties of these sequence spaces and obtain some inclusion relations.

Key Words: Entire sequence, Analytic sequence, Invariant mean, Semi norm.

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1. Introduction

A complex sequence, whose $k^{th}$ term is $x_k$, is denoted by $\{x_k\}$ or simply $x$. Let $\phi$ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^s < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$.

A sequence $x$ is called entire sequence if $\lim_{k \to \infty} |x_k|^s = 0$. The vector space of all entire sequences will be denoted by $\Gamma$.

Let $\sigma$ be a one-one mapping of the set of positive integers into itself such that $\sigma m(n) = \sigma(\sigma m(n) - 1)$, $m = 1, 2, 3, \ldots$.

A continuous linear functional $\phi$ on $\Lambda$ is said to be an invariant mean or a $\sigma$-mean if and only if (1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n$ (2) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and (3) $\phi(\{x_n(n)\}) = \phi(\{x_n\})$ for all $x \in \Lambda$. For certain kinds of mappings $\sigma$, every invariant mean $\phi$ extends the limit functional on the space $C$ of all real convergent sequences in the sense that $\phi(x) = \lim_{n \to \infty} x_n$ for all $x \in C$. Consequently $C \subset V_\sigma$, where $V_\sigma$ is the set of analytic sequences all of those $\sigma$-means are equal.

If $x = (x_n)$, set $Tx = (T_x)^{1/n} = (x_\sigma(n))$. It can be shown that $V_\sigma = \{x = (x_n) : m \lim_{n \to \infty} t_{mn}(x_n)^{1/n} = L \text{ uniformly in } n, L = \sigma - n \lim_{n \to \infty} (x_n)^{1/n}\}$

where

$$t_{mn}(x) = \frac{(x_n + T_{mx} + \cdots + T^m x_n)^{1/n}}{m + 1} \quad (1)$$

Given a sequence $x = \{x_k\}$ its $n^{th}$ section is the sequence $x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$, $\delta^{(n)} = \{0, 0, \ldots, 1, 0, 0, \ldots\}$, 1 in the $n^{th}$ place and zeros elsewhere. An FK-space

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2. Definitions and Preliminaries

Definition 2.1 The space consisting of all those sequences \( x \) in \( w \) such that \( \left( |x_k|^{1/k} \right) \rightarrow 0 \) as \( k \rightarrow \infty \) is denoted by \( \Gamma \). In other words \( \left( |x_k|^{1/k} \right) \) is a null sequence. \( \Gamma \) is called the space of entire sequences. The space \( \Gamma \) is a metric space with the metric \( d(x, y) = \left\{ \sup_k \left( |x_k - y_k|^{1/k} \right) : k = 1, 2, 3, \ldots \right\} \) for all \( x = \{x_k\} \) and \( y = \{y_k\} \) in \( \Gamma \).

Definition 2.2 The space consisting of all those sequences \( x \) in \( w \) such that \( \left( \sup_k \left( |x_k|^{1/k} \right) \right) < \infty \) is denoted by \( \Lambda \). In other words \( \left( \sup_k \left( |x_k|^{1/k} \right) \right) \) is a bounded sequence.

Definition 2.3 Let \( p, q \) be semi norms on a vector space \( X \). Then \( p \) is said to be stronger than \( q \) if whenever \( (x_n) \) is a sequence such that \( p(x_n) \rightarrow 0 \), then also \( q(x_n) \rightarrow 0 \). If each is stronger than the other, then \( p \) and \( q \) are said to be equivalent.

Lemma 2.4 Let \( p \) and \( q \) be semi norms on a linear space \( X \). Then \( p \) is stronger than \( q \) if and only if there exists a constant \( M \) such that \( q(x) \leq Mp(x) \) for all \( x \in X \).

Definition 2.5 A sequence space \( E \) is said to be solid or normal if \( (\alpha_k x_k) \in E \) whenever \( (x_k) \in E \) and for all sequences of scalars \( (\alpha_k) \) with \( |\alpha_k| \leq 1 \), for all \( k \in N \).

Definition 2.6 A sequence space \( E \) is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.7 From the above two definitions, it is clear that a sequence space \( E \) is solid implies that \( E \) is monotone.

Definition 2.8 A sequence \( E \) is said to be convergence free if \( (y_k) \in E \) whenever \( (x_k) \in E \) and \( x_k = 0 \) implies that \( y_k = 0 \).

Let \( p = (p_k) \) be a sequence of positive real numbers with \( 0 < p_k < \sup p_k = G \). Let \( D = \text{Max}(1, 2^{G-1}) \). Then for \( a_k, b_k \in C \), the set of complex numbers for all \( k \in N \) we have
\[
|a_k + b_k|^{1/k} \leq D \left\{ |a_k|^{1/k} + |b_k|^{1/k} \right\}.
\]

(2)

Let \((X, q)\) be a semi normed space over the field \( C \) of complex numbers with the semi norm \( q \). The symbol \( \Lambda(X) \) denotes the space of all analytic sequences defined over \( X \). We define the following sequence spaces:
\[
\Lambda(p, \sigma, q, s) = \left\{ x \in \Lambda(X): \sup_{n,k} k^{-s} \left[q \left( |x_{\sigma(n)}|^{1/k} \right) \right]^{p_k} < \infty \text{uniformly in } n \geq 0, s \geq 0 \right\}
\]
Theorem 3.1 \( \Gamma(p, \sigma, q, s) \) is a linear space over the set of complex numbers.

Proof: The proof is easy, so omitted.

Theorem 3.2 \( \Gamma(p, \sigma, q, s) \) is a paranormed space with

\[
g^*(x) = \left\{ \sup_{k \geq 1} k^{-s} \left[ q \left( |x_{\sigma^k(n)}|^{1/k} \right) \right] \text{ uniformly in } n > 0 \right\}
\]

where \( H = \max(1, \sup_k p_k) \).

Proof: Clearly \( g(x) = g(-x) \) and \( g(\theta) = 0 \), where \( \theta \) is the zero sequence. It can be easily verified that \( g(x + y) \leq g(x) + g(y) \). Next \( x \to \theta, \lambda \) fixed implies \( g(\lambda x) \to 0 \). Also \( x \to \theta \) and \( \lambda \to 0 \) implies \( g(\lambda x) \to 0 \). The case \( \lambda \to 0 \) and \( x \) fixed implies that \( g(\lambda x) \to 0 \) follows from the following expressions.

\[
g(\lambda x) = \left\{ (|\lambda| r)^{p_m/H} \sup_{k \geq 1} k^{-s} \left[ q \left( |x_{\sigma^k(n)}|^{1/k} \right) \right] , \text{ uniformly in } n, m \in N \right\}
\]

where \( r = \frac{1}{|\lambda|} \). Hence \( \Gamma(p, \sigma, q, s) \) is a paranormed space. This completes the proof.

Theorem 3.3 \( \Gamma(p, \sigma, q, s) \cap \Lambda(p, \sigma, q, s) \subseteq \Gamma(p, \sigma, q, s) \).

Proof: The proof is easy, so omitted.

Theorem 3.4 \( \Gamma(p, \sigma, q, s) \subset \Lambda(p, \sigma, q, s) \).

Proof: The proof is easy, so omitted.

Remark 3.1 Let \( q_1 \) and \( q_2 \) be two semi norms on \( X \), we have

(i) \( \Gamma(p, \sigma, q_1, s) \cap \Gamma(p, \sigma, q_2, s) \subseteq \Gamma(p, \sigma, q_1 + q_2, s) \);

(ii) If \( q_1 \) is stronger than \( q_2 \), then \( \Gamma(p, \sigma, q_1, s) \subseteq \Gamma(p, \sigma, q_2, s) \);

(iii) If \( q_1 \) is equivalent to \( q_2 \), then \( \Gamma(p, \sigma, q_1, s) = \Gamma(p, \sigma, q_2, s) \).

Theorem 3.5 (i) Let \( 0 \leq p_k \leq r_k \) and \( \left\{ \frac{r_k}{p_k} \right\} \) be bounded. Then \( \Gamma(r, \sigma, q, s) \subset \Gamma(p, \sigma, q, s) \);

(ii) \( s_1 \leq s_2 \) implies \( \Gamma(p, \sigma, q, s_1) \subset \Gamma(p, \sigma, q, s_2) \).
Proof of (i):

Let

\[ x \in \Gamma(r, \sigma, q, s) \]  

(3)

\[ k^{-s} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{r_k} \right] \rightarrow 0 \text{ as } k \rightarrow \infty \]  

(4)

Let \( t_k = k^{-s} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{r_k} \right] \) and \( \lambda_k = \frac{p_k}{r_k} \). Since \( p_k \leq r_k \), we have \( 0 \leq \lambda_k \leq 1 \).

Take \( 0 < \lambda > \lambda_k \). Define \( u_k = t_k(t_k \geq 1); u_k = 0(t_k < 1) \); and \( v_k = 0(t_k \geq 1); v_k = t_k(t_k < 1) \); \( t_k = u_k + v_k \); \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \). Now it follows that

\[ u_k^{\lambda_k} \leq t_k \text{ and } v_k^{\lambda_k} \leq v_k \]  

(5)

\[(i.e) \ t_k^{\lambda_k} \leq t_k + v_k^{\lambda_k} \text{ by (5)} \]

\[ k^{-s} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{r_k} \right]^{p_k/r_k} \leq k^{-s} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{r_k} \right]^{k^{-s}} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{r_k} \right]^{p_k} \leq k^{-s} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{r_k} \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ by (4)} . \]

Hence

\[ x \in \Gamma(p, \sigma, q, s) \]  

(6)

From (3) and (6) we get \( \Gamma(r, \sigma, q, s) \subset \Gamma(p, \sigma, q, s) \). This completes the proof.

Proof of (ii): The proof is easy, so omitted.

Theorem 3.6 The space \( \Gamma(p, \sigma, q, s) \) is solid and as such is monotone.

Proof: Let \( (x_k) \in \Gamma(p, \sigma, q, s) \) and \( (\alpha_k) \) be a sequence of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in N \). Then \( k^{-s} \left[ q \left( \left| \alpha_k x_{\sigma^k(n)} \right|^{1/k} \right)^{p_k} \right] \leq k^{-s} \left[ q \left( \left| x_{\sigma^k(n)} \right|^{1/k} \right)^{p_k} \right] \) for all \( k \in N \). This completes the proof.

Theorem 3.7 The space \( \Gamma(p, \sigma, q, s) \) are not convergence free in general.

Proof: The proof follows from the following example.

Example: Let \( s = 0; p_k = 1 \) for \( k \) even and \( p_k = 2 \) for \( k \) odd. Let \( X = C, q(x) = |x| \) and \( \sigma(n) = n + 1 \) for all \( n \in N \). Then we have \( \sigma^2(n) = \sigma(\sigma(n)) = \sigma(n + 1) = (n + 1) + 1 = n + 2 \) and \( \sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n + 2) = (n + 2) + 1 = n + 3 \). Therefore \( \sigma^k(n) = (n + k) \) for all \( n, k \in N \). Consider the sequences \( (x_k) \) and \( (y_k) \) defined as \( x_k = \left( \frac{1}{2} \right)^k \) and \( y_k = k^k \) for all \( k \in N \). (i.e) \( |x_k|^{1/k} = \frac{1}{2}^k \) and \( |y_k|^{1/k} = k \) for all \( k \in N \).

Hence \( \left( \frac{1}{(n+k)} \right)^{n+k} \rightarrow 0 \text{ as } k \rightarrow \infty \). Therefore \( (x_k) \in \Gamma(p, \sigma) \). But \( \left( \frac{1}{(n+k)} \right)^{n+k} \rightarrow r \) \( r \neq 0 \) as \( k \rightarrow \infty \). Hence \( (y_k) \notin \Gamma(p, \sigma) \). Hence the space \( \Gamma(p, \sigma, q, s) \) are not convergence free in general. This completes the proof.
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**References**


N.Subramanian
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.
E-mail address: nsmaths@yahoo.com

and

K.Chandrasekhara Rao
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.
E-mail address: kchandrasekhara@rediffmail.com

and

K.Balasubramanian
Department of Mathematics, SASTRA University,
Thanjavur-613 401, India.
E-mail address: k_bala27@yahoo.co.in