Existence of solutions for a fourth order problem at resonance

El. M. Hssini, M. Massar, M. Talbi and N. Tsouli

ABSTRACT: In this work, we are interested at the existence of nontrivial solutions of two fourth order problems governed by the weighted p-biharmonic operator. The first is the following

\[
\Delta(\rho|\Delta u|^{p-2} \Delta u) = \lambda_1 m(x)|u|^{p-2} u + f(x, u) - h \quad \text{in} \quad \Omega, \\
u = \Delta u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \(\lambda_1\) is the first eigenvalue for the eigenvalue problem \(\Delta(\rho|\Delta u|^{p-2} \Delta u) = \lambda m(x)|u|^{p-2} u \quad \text{in} \quad \Omega, \quad u = \Delta u = 0 \quad \text{on} \quad \partial \Omega\). In the second problem, we replace \(\lambda_1\) by \(\lambda\) such that \(\lambda_1 < \lambda < \bar{\lambda}\), where \(\bar{\lambda}\) is given below.

Key Words: p-biharmonic, weight, resonance, saddle point theorem.

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1. Introduction and main results

In the present paper, we are concerned with the existence of weak solutions of the following problem

\[
\left\{ \begin{array}{ll}
\Delta(\rho|\Delta u|^{p-2} \Delta u) = \lambda_1 m(x)|u|^{p-2} u + f(x, u) - h & \text{in} \quad \Omega, \\
u = \Delta u = 0 & \text{on} \quad \partial \Omega,
\end{array} \right.
\]

(1.1)

where \(p > 1\), \(\Omega\) is a bounded domain of \(\mathbb{R}^N (N \geq 1)\) with smooth boundary \(\partial \Omega\), \(\rho \in C(\overline{\Omega})\), with min\(\rho(x) > 0\), \(f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) is a bounded Carathéodory function, \(h \in L^p(\Omega)\), \(\left(\frac{p'}{p-1}\right)\), \(m \in C(\overline{\Omega})\) is nonnegative weight function and \(\lambda_1\) design the first eigenvalue for the eigenvalue problem

\[
\left\{ \begin{array}{ll}
\Delta(\rho|\Delta u|^{p-2} \Delta u) = \lambda m(x)|u|^{p-2} u & \text{in} \quad \Omega, \\
u = \Delta u = 0 & \text{on} \quad \partial \Omega.
\end{array} \right.
\]

(1.2)

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, see for example [1,3,6,7,12]. In [7], Liu and Squassina study the following p-biharmonic problem

\[
\left\{ \begin{array}{ll}
\Delta(|\Delta u|^{p-2} u) = g(x, u) & \text{in} \quad \Omega, \\
u = \Delta u = 0 & \text{on} \quad \partial \Omega.
\end{array} \right.
\]

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Under some conditions on \( g(x, u) \) at resonance, the authors established the existence of at least one nontrivial solution.

According to the work of Talbi and Tsouli \([10]\), the eigenvalue problem (1.2) has a nondecreasing and unbounded sequence of eigenvalues, and the first eigenvalue \( \lambda_1 \) is given by

\[
\lambda_1 = \inf_{u \in X} \left\{ \int_{\Omega} \rho |\Delta u|^p dx : \int_{\Omega} m(x)|u|^p dx = 1 \right\},
\]

where \( X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) is the reflexive Banach space endowed with the norm

\[
||u|| = \left( \int_{\Omega} \rho |\Delta u|^p dx \right)^{1/p}.
\]

Since \( m \in C(\Omega) \) and \( m \geq 0 \), \( \lambda_1 \) is positive, simple and isolated. Therefore

\[
\int_{\Omega} \rho |\Delta u|^p dx \geq \lambda_1 \int_{\Omega} m(x)|u|^p dx \quad \text{for all } u \in X.
\]

(1.3)

Moreover, there exists a unique positive eigenfunction \( \varphi_1 \) associated to \( \lambda_1 \), which can be chosen normalized. Let \( \lambda_2 := \inf \{ \lambda : \lambda \text{ is a eigenvalue of (1.2), with } \lambda > \lambda_1 \} \).

The fact that \( \lambda_1 \) is isolated implies that \( \lambda_1 < \lambda_2 \). It can also be shown (see Lemma 2.1) that there exists \( \lambda \in (\lambda_1, \lambda_2) \) such that

\[
\int_{\Omega} \rho |\Delta u|^p dx \geq \lambda \int_{\Omega} m(x)|u|^p dx,
\]

(1.4)

for all \( u \in X \) with \( \int_{\Omega} m(x)\varphi_1^{p-1} u dx = 0 \).

In addition, we study the existence of solutions for the following boundary value problem

\[
\begin{cases}
\Delta(|\Delta u|^{p-2} \Delta u) = \lambda m(x)|u|^{p-2} u + f(x, u) - h & \text{in } \Omega \\
\Delta u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.5)

We assume that the function \( f \) satisfy the following hypothesise:

\((H)\) For almost every \( x \in \Omega \), there exist

\[
\lim_{s \to -\infty} f(x, s) = l(x), \quad \lim_{s \to +\infty} f(x, s) = k(x).
\]

(1.6)

Let us recall the minimum principle and the saddle point theorem (see \([9]\)).

**Theorem 1.1.** Let \( X \) be a Banach space and \( \Phi \in C^1(X, \mathbb{R}) \). Assume that

(i) \( \Phi \) satisfies the Palais-Smale condition,

(ii) \( \Phi \) is bounded from below \( c = \inf_X \Phi \).

Then there exists \( u_0 \in X \) such that \( \Phi(u_0) = c \).
Theorem 1.2. Let $X$ be a Banach space. Let $\Phi : X \to \mathbb{R}$ be a $C^1$ functional that satisfies the Palais-Smale condition, and suppose that $X = V \oplus W$, with $V$ a finite dimensional subspace of $X$. If there exists $R > 0$ such that
\[
\max_{v \in V, ||v|| = R} \Phi(v) < \inf_{w \in W} \Phi(w),
\]
then $\Phi$ has at least a critical point on $X$.

Now, we are ready to state our main results.

Theorem 1.3. Assume that (1.6) holds. Suppose that $h \in L^p' (\Omega)$ is such that either
\[
\int_{\Omega} k(x) \varphi_1 dx < \int_{\Omega} h(x) \varphi_1 dx < \int_{\Omega} l(x) \varphi_1 dx \quad (1.7)
\]
or
\[
\int_{\Omega} l(x) \varphi_1 dx < \int_{\Omega} h(x) \varphi_1 dx < \int_{\Omega} k(x) \varphi_1 dx. \quad (1.8)
\]
Then problem (1.1) has at least a weak solution.

Theorem 1.4. Assume that (1.6) holds. If $h \in L^p' (\Omega)$ satisfy (1.7) or (1.8), then problem (1.5) with $\lambda_1 < \lambda < \lambda$, has at least one solution.

2. Preliminaries and proofs of Theorems

We consider the following energy functional $\Phi : X \to \mathbb{R}$ defined by
\[
\Phi(u) = \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x) |u|^p dx + \int_{\Omega} F(x, u) dx + \int_{\Omega} h u dx,
\]
where
\[
F(x, t) = \int_0^t f(x, s) ds \quad \text{for almost every } x \in \Omega, \forall t \in \mathbb{R}.
\]
It is well known that $\Phi \in C^1(X, \mathbb{R})$, with derivative at point $u \in X$ is given by
\[
\langle \Phi'(u), v \rangle = \int_{\Omega} \rho |\Delta u|^{p-2} \Delta u \Delta v dx - \lambda_1 \int_{\Omega} m(x) |u|^{p-2} u v dx - \int_{\Omega} f(x, u) v dx + \int_{\Omega} h v dx,
\]
for every $v \in X$.

Let denote $V = \langle \varphi_1 \rangle$ the linear spans of $\varphi_1$ and
\[
W = \left\{ u \in X : \int_{\Omega} m(x) \varphi_1^{p-1} u dx = 0 \right\}. \quad (2.1)
\]
Then we can decompose $X$ as a direct sum of $V$ and $W$. In fact, let $u \in X$, writing
\[
u = \alpha \varphi_1 + w,
\]
where $w \in X$, and $\alpha = \lambda_1 \int_{\Omega} m(x) \varphi_1^{p-1} u dx$.

Since
\[
\int_{\Omega} \rho |\Delta \varphi_1|^p dx = 1,
\]
\[
\int_{\Omega} m(x) \varphi_1^{p-1} w dx = 0.
\]

Therefore \( w \in W \), hence \( X = V \oplus W \).

We begin by establishing the existence of \( \overline{\lambda} \) for which (1.4) holds.

**Lemma 2.1.** There exists \( \overline{\lambda} \in (\lambda_1, \lambda_2] \) such that

\[
\int_{\Omega} \rho |\Delta u|^p dx \geq \overline{\lambda} \int_{\Omega} m(x)|u|^p dx,
\]

(2.2)

for all \( u \in W \).

**Proof:** Let

\[
\lambda = \inf \left\{ \int_{\Omega} \rho |\Delta u|^p dx : u \in W, \int_{\Omega} m(x)|u|^p dx = 1 \right\}.
\]

This value is attained in \( W \). To see why this is so, let \( (u_n) \) be a sequence in \( W \), satisfying \( \int_{\Omega} m(x)|u_n|^p dx = 1 \) for all \( n \), and \( \int_{\Omega} \rho |\Delta u_n|^p dx \to \lambda \). It follows that \( (u_n) \) is bounded in \( X \) and therefore, up to subsequence, we may assume that

\[
u_n \to u \text{ weakly in } X \quad \text{and} \quad u_n \to u \text{ strongly in } L^p(\Omega).
\]

From the strong convergence of the sequence in \( L^p(\Omega) \) we obtain

\[
\int_{\Omega} m(x)|u|^p dx = \lim_{n \to \infty} \int_{\Omega} m(x)|u_n|^p dx = 1
\]

and

\[
\int_{\Omega} m(x) \varphi_1^{p-1} u dx = \lim_{n \to \infty} \int_{\Omega} m(x) \varphi_1^{p-1} u_n dx = 0,
\]

so that \( u \in W \). By the weakly lower semicontinuity of the norm \(||.||\), we get

\[
\lambda \leq \int_{\Omega} \rho |\Delta u|^p dx \leq \liminf_{n \to \infty} \int_{\Omega} \rho |\Delta u_n|^p dx = \lambda,
\]

and hence \( \lambda \) is attained at \( u \).

Now we claim that \( \lambda > \lambda_1 \). It follows from (1.3) that \( \lambda \geq \lambda_1 \). If \( \lambda = \lambda_1 \), by simplicity of \( \lambda_1 \) there is \( \alpha \in \mathbb{R} \) such that \( u = \alpha \varphi_1 \). Since \( u \in W \),

\[
\alpha \int_{\Omega} m(x) \varphi_1^p dx = 0,
\]

which implies \( \alpha = 0 \). This contradicts the fact that \( \int_{\Omega} m(x)|u|^p dx = 1 \). So, choose \( \overline{\lambda} = \min\{\lambda, \lambda_2\} \). It is clear that \( \overline{\lambda} \) satisfies (2.2) and the proof of lemma is complete.

\( \square \)
Lemma 2.2. Assume that (1.6) and (1.7) or (1.8) are verified. Then the functional \( \Phi \) satisfies the Palais-Smale condition on \( X \).

Proof: Let \((u_n)\) be a sequence in \( X \), and \( c \) a real number such that:

\[
|\Phi(u_n)| \leq c \quad \text{for all } n, \tag{2.3}
\]

\[
\Phi'(u_n) \to 0. \tag{2.4}
\]

We claim that \((u_n)\) is bounded in \( X \). Indeed, suppose by contradiction that

\[
||u_n|| \to +\infty, \quad \text{as } n \to +\infty.
\]

Put \( v_n = u_n/||u_n|| \), thus \((v_n)\) is bounded, for a subsequence still denoted \((v_n)\), we can assume that \( v_n \to v \) weakly in \( X \), by Sobolev injection theorem we have \( v_n \to v \) strongly in \( L^p(\Omega) \), and \( v_n \to v \) a.e. in \( \Omega \). Dividing (2.3) by \( ||u_n||^p \), we get

\[
\lim_{n \to +\infty} \left( \frac{1}{p} \int_{\Omega} \rho |\Delta v_n|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x)|v_n|^p dx - \int_{\Omega} \frac{F(x, u_n)}{||u_n||^p} dx + \int_{\Omega} h \frac{u_n}{||u_n||^p} dx \right) = 0. \tag{2.5}
\]

By the hypotheses on \( f, h \) and \((u_n)\), we obtain

\[
\lim_{n \to +\infty} \int_{\Omega} \frac{F(x, u_n)}{||u_n||^p} dx = \lim_{n \to +\infty} \int_{\Omega} h \frac{u_n}{||u_n||^p} dx = 0,
\]

while

\[
\lim_{n \to +\infty} \int_{\Omega} m(x)|v_n|^p dx = \int_{\Omega} m(x)|v|^p dx
\]

then, from (2.5) we deduce that

\[
1 = \lim_{n \to +\infty} \int_{\Omega} \rho |\Delta v_n|^p dx = \lambda_1 \int_{\Omega} m(x)|v|^p dx.
\]

Then \( v \not\equiv 0 \). According to the definition of \( \lambda_1 \) and the weak lower semi continuity of norm, one has

\[
\lambda_1 \int_{\Omega} m(x)|v|^p dx \leq \int_{\Omega} \rho |\Delta v|^p dx \leq \liminf_{n \to +\infty} \int_{\Omega} \rho |\Delta v_n|^p dx = \lambda_1 \int_{\Omega} m(x)|v|^p dx.
\]

This implies that

\[
v_n \to v \text{ strongly in } X \quad \text{and} \quad \int_{\Omega} \rho |\Delta v|^p dx = \lambda_1 \int_{\Omega} m|v|^p dx.
\]

By the definition of \( \varphi_1 \), we deduce that \( v = \pm \varphi_1 \).

On the other hand, from (2.3) we have

\[
-cp \leq \int_{\Omega} \rho |\Delta u_n|^p dx - \lambda_1 \int_{\Omega} m(x)|u_n|^p dx - p \int_{\Omega} F(x, u_n) dx + p \int_{\Omega} h u_n dx \leq cp \tag{2.6}
\]
In view of (2.4), for all \( \varepsilon > 0 \) and \( n \) large enough, we have
\[
-\varepsilon \|u_n\| \leq -\int_\Omega \rho |\Delta u_n|^p dx + \lambda_1 \int_\Omega m|u_n|^p dx + \int_\Omega f(x, u_n) u_n dx - \int_\Omega h u_n dx \leq \varepsilon \|u_n\|
\]
(2.7)

Let
\[
g(x, s) = \begin{cases} f(x, s) & \text{if } s \neq 0 \\ f(x, 0) & \text{if } s = 0. \end{cases}
\]
(2.8)

Suppose that \( v_n \rightarrow -\varphi_1 \) (for example), then \( u_n(x) \rightarrow -\infty \) for a.e. \( x \in \Omega \), it follows from (1.6) that
\[
\begin{cases} f(x, u_n) \rightarrow l(x) & \text{a.e } x \in \Omega \\ g(x, u_n) \rightarrow l(x) & \text{a.e } x \in \Omega, \end{cases}
\]
Moreover, the Lebesgue's theorem imply
\[
\lim_{n \to +\infty} \int_\Omega (f(x, u_n)v_n - pg(x, u_n)v_n) dx = (p - 1) \int_\Omega l(x)\varphi_1 dx. \tag{2.9}
\]

Combining (2.6) and (2.7), we get
\[
-cp - \varepsilon \|u_n\| \leq \int_\Omega f(x, u_n) u_n dx - p \int_\Omega F(x, u_n) dx + (p - 1) \int_\Omega h u_n dx \leq cp + \varepsilon \|u_n\|.
\]

Dividing by \( \|u_n\| \) the last inequalities, we obtain
\[
\frac{-cp}{\|u_n\|} - \varepsilon \leq \int_\Omega f(x, u_n) v_n dx - p \int_\Omega g(x, u_n) v_n dx + (p - 1) \int_\Omega h v_n dx \leq \frac{cp}{\|u_n\|} + \varepsilon,
\]
and passing to the limits, we deduce from (2.9) that
\[
\int_\Omega l(x)\varphi_1 dx = \int_\Omega h(x)\varphi_1 dx,
\]
which contradicts both (1.7) and (1.8). Thus \( (u_n) \) is bounded in \( X \), for a subsequence denoted also \( (u_n) \), there exists \( u \in X \) such that \( u_n \rightharpoonup u \) weakly in \( X \), and strongly in \( L^p(\Omega) \). From
\[
\lim_{n \to +\infty} \langle \Phi'(u_n), (u_n - u) \rangle = 0,
\]
that is
\[
\langle \Phi'(u_n), (u_n - u) \rangle = \int_\Omega \rho |\Delta u_n|^{p-2} \Delta u_n \Delta (u_n - u) dx
\]
\[
- \lambda_1 \int_\Omega m(x)|u_n|^{p-2} u_n (u_n - u) dx
\]
\[
- \int_\Omega f(x, u_n)(u_n - u) dx + \int_\Omega h(u_n - u) dx
\]
\[
= o_n(1).
\]
Using the hypotheses on \( m, h \) and \( f \), we see that

\[
\lim_{n \to +\infty} \int_{\Omega} m(x) |u_n|^{p-2} u_n (u_n - u) dx = 0, \quad \lim_{n \to +\infty} \int_{\Omega} f(x, u_n) (u_n - u) dx = 0
\]

\[
\lim_{n \to +\infty} \int_{\Omega} h (u_n - u) dx = 0.
\]

Consequently,

\[
\lim_{n \to +\infty} \int_{\Omega} \rho |\Delta u_n|^{p-2} \Delta u_n \Delta (u_n - u) dx = 0.
\]

In the same way, we obtain

\[
\lim_{n \to +\infty} \int_{\Omega} \rho |\Delta u|^{p-2} \Delta u \Delta (u_n - u) dx = 0.
\]

Therefore

\[
0 = \lim_{n \to +\infty} \int_{\Omega} \left( \rho |\Delta u_n|^{p-2} \Delta u_n - \rho |\Delta u|^{p-2} \Delta u \right) (u_n - u) dx
\]

\[
\geq \lim_{n \to +\infty} (|| u_n ||^{p-1} - || u ||^{p-1}) (|| u_n || - || u ||) \geq 0,
\]

hence \( || u_n || \to || u || \). By the uniform convexity of \( X \), it follows that \( u_n \to u \) strongly in \( X \) and \( \Phi \) satisfies the \((PS)\) condition.

\[\square\]

**Lemma 2.3.** Assume that (1.6) and (1.7) are satisfied. Then the functional \( \Phi \) is coercive on \( X \).

**Proof:** Suppose by contradiction that \( \Phi \) is not coercive, then there exists a sequence \( (u_n) \) such that \( || u_n || \to +\infty \), and \( |\Phi(u_n)| \leq C \).

In the proof of lemma 2.2, we have showed that \( v_n = u_n / || u_n || \to \pm \varphi_1 \).

Since

\[
0 \leq \int_{\Omega} \rho |\Delta u_n|^{p} dx - \lambda_1 \int_{\Omega} m |u_n|^{p} dx
\]

\[
- \int_{\Omega} F(x, u_n) dx + \int_{\Omega} h u_n dx \leq \Phi(u_n) \leq C. \quad (2.10)
\]

Assume \( v_n \to -\varphi_1 \) (for example). Dividing (2.10) by \( || u_n || \), we get

\[
- \int_{\Omega} \frac{F(x, u_n)}{|| u_n ||} dx + \int_{\Omega} \frac{h u_n}{|| u_n ||} dx \leq \frac{C}{|| u_n ||}.
\]

Passing to the limits, we have

\[
\int_{\Omega} l(x) \varphi_1 dx \leq \int_{\Omega} h(x) \varphi_1 dx
\]

which contradicts (1.7).

\[\square\]
Proof: [Proof of Theorem 1.3]. If (1.7) holds, the coerciveness of the functional \( \Phi \) and the Palais-Smale condition entain, from theorem 1.1, that \( \Phi \) attains its minimum, so problem (1.1) admits at least a weak solution in \( X \).

If (1.8) holds, then \( \Phi \) has the geometry of the saddle point theorem 1.2. Indeed, splitting \( X = V \oplus W \). Let \( u \in W \), using Höder inequality and the properties of \( F \), since \( \lambda > \lambda_1 \)

\[
\Phi(u) \geq \frac{1}{p} \int_{\Omega} \rho|\Delta u|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x)|u|^p dx - \int_{\Omega} F(x, u) dx + \int_{\Omega} h(x) u dx
\]

\[
\geq \frac{1}{p} \left( 1 - \frac{\lambda_1}{\lambda} \right) ||u||^p - C(||b||_\infty ||\cdot|| + ||h||_{p'}) ||u||, \tag{2.11}
\]

where \( C \) is the embedding constants of Sobolev, \( ||\cdot||_{p'} \) and \( ||\cdot||_\infty \) denote the norms in \( L^{p'}(\Omega) \) and \( L^{\infty}(\Omega) \) respectively. Then \( \Phi \) is bounded from below on \( W \), is a consequence of the assumption that \( p > 1 \), so that

\[
\inf_{w \in W} \Phi(w) > -\infty. \tag{2.12}
\]

On the other hand, for every \( t \in \mathbb{R} \), one has

\[
\Phi(t\varphi_1) = -\int_{\Omega} F(x, t\varphi_1) dx + t \int_{\Omega} h(x) \varphi_1 dx
\]

\[
= t \left( \int_{\Omega} h(x) \varphi_1 dx - \int_{\Omega} g(x, t\varphi_1) \varphi_1 dx \right)
\]

where \( g \) has been defined by (2.8). From the Lebesgue theorem, it follows that

\[
\lim_{t \to +\infty} \left( \int_{\Omega} h(x) \varphi_1 dx - \int_{\Omega} g(x, t\varphi_1) \varphi_1 dx \right) = \int_{\Omega} (h(x) - k(x)) \varphi_1 dx, \tag{2.13}
\]

and the limit is negative by (1.8). Analogously, if \( t \) tends to \(-\infty\), we have the same result with \( k(x) \) exchanged with \( l(x) \), so that the limit is positive by (1.8). In both cases we get

\[
\lim_{t \to \pm \infty} \Phi(t\varphi_1) = -\infty \tag{2.14}
\]

By (2.12) and (2.14), there exists \( R > 0 \) such that

\[
\max_{v \in V, ||v|| = R} \Phi(v) < \inf_{w \in W} \Phi(w).
\]

Hence, \( \Phi \) satisfies the hypotheses of Theorem 1.2, and there exists a critical point of \( \Phi \), that is a solution of (1.1).

\[\square\]

Proof: [Proof of Theorem 1.4]. The result of lemma 2.2 holds true for the Euler functional associated to problem (1.5), that is

\[
\Phi_\lambda(u) = \frac{1}{p} \int_{\Omega} \rho|\Delta u|^p dx - \frac{\lambda}{p} \int_{\Omega} m(x)|u|^p dx - \int_{\Omega} F(x, u) dx + \int_{\Omega} h u dx \tag{2.15}
\]
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for every \( u \in X \). Indeed, let \((u_n)\) be a sequence satisfying (2.3) and (2.4), suppose that \((u_n)\) is unbounded, and define \( v_n = u_n/\|u_n\| \), so that, up to subsequence, \((v_n)\) converges weakly to a function \( v \) in \( X \). Dividing (2.4) by \( \|u_n\|^{p-1} \), and then taking \( \langle \Phi'_{\lambda}(u_n), v_n - v \rangle = o_n(1) \), we get

\[
\lim_{n \to +\infty} \int_{\Omega} \rho|\Delta v_n|^{p-2} \Delta v_n \Delta (v_n - v) \, dx = 0
\]

this fact implies (as in proof of lemma 2.2) that \( v_n \to v \) strongly in \( X \). Since

\[
\int_{\Omega} \rho|\Delta v|^{p-2} \Delta v \Delta \psi \, dx = \lambda \int_{\Omega} m|v|^{p-2} v \psi \, dx,
\]

so that \( v \) solve the problem \( \Delta (\rho|\Delta u|^{p-2} \Delta u) = \lambda m(x)|u|^{p-2} u \) with Navier boundary condition on \( \partial \Omega \). But this equation, being \( \lambda \in (\lambda_1, \lambda) \subset (\lambda_1, \lambda_2) \), has zero as the only solution by definition of \( \lambda \). Thus \( v = 0 \), a contradiction with the strong convergence of \( v_n \) to \( v \). Hence \((u_n)\) is bounded. This implies, by same argument in proof of lemma 2.2, that \((u_n)\) is strongly convergent.

On the other hand, as in the second part of the proof of Theorem 1.3, rewrite everything with \( \lambda \) instead of \( \lambda_1 \) and use the fact that \( \lambda > \lambda_1 \) and \( p > 1 \), we have, as before

\[
\lim_{t \to \pm \infty} \Phi_{\lambda}(t \varphi_1) = -\infty.
\]

Using again the saddle point theorem, the desired result follows.

\[ \square \]

References


*El. Miloud Hssini; Mohammed Massar; Najib Tsouli*

*University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco. E-mail address: hssini1975@yahoo.fr; massarmed@hotmail.com; tsouli@hotmail.com* and

*Mohamed Talbi*

*Centre Régional de Métiers de l’Éducation et de Formation (CRMEF), Oujda, Morocco. E-mail address: talbi_md@yahoo.fr*