Derivation on Vinberg Rings

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ABSTRACT: A nonassociative ring which contains a well-known associative ring or left symmetric ring also known as Vinberg ring is of great interest. A method to construct Vinberg nonassociative ring is given; Vinberg nonassociative ring $V_N^{n,m,s}$ is shown as simple; all the derivations of nonassociative simple Vinberg algebra defined are determined; and finally in solid algebra it is shown that if $\theta$ is a nonzero endomorphism of $V_{N0,0,1}$, then $\theta$ is an epimorphism.

Key Words: Nonassociative ring, Simple, Vinberg ring, Derivation, Solid algebra.

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1. Preliminaries

Let $(A, *, +)$ be a nonassociative algebra then the antisymmetrized algebra $(A^-, [\cdot,\cdot], +)$ with the same set $A$ and the Lie bracket $[,]$ is defined as follows: $[x, y] = x * y - y * x$ for any $x, y \in A^-$. Choi proposed an interesting problem [9]: Does the equality $Aut_F(A) = Aut_{Lie}(A^-)$ hold? The answer is no generally. Any derivation of an algebra $A$ is a derivation of the antisymmetrized algebra $A^-$. He also proposed an interesting problem: Is $Der(A) = Der_{Lie}(A^-)$? If $\theta$ is an automorphism of Vinberg ring $VN$ then the $Der(VN)$ is also an automorphism. For a $p$-torsion free Vinberg algebra, we do not know $Der(A)$ generally. Our method of finding $Der(V_{N0,0,1})$ will give a good modification to find $Der(A)$ of an algebra $A$. The authors have given the description of a 2-torsion free Vinberg ($-1,1$) ring $R$ in [2]. They have shown that if every nonzero root space of $R^-$ for $S$ is one-dimensional where $S$ is a split abelian Cartan subring of $R^-$ which is nil on $R$ then $R$ is a Lie ring isomorphic to $R^-$. In this paper we extend the results of [2]

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to $\overline{VN_{0,0,1}}$ algebra. A nonzero endomorphism of $\overline{VN_{0,0,1}}$ is an epimorphism. A nonassociative ring $R$ is called a Vinberg ring if it satisfies the identity

$$(x, y, z) = (y, x, z)$$

(1.1)

where $(x, y, z) = (xy)z - x(yz)$ for $x, y, z \in R$. Throughout this paper $Z$ and $N$ are the sets of integers and non-negative integers respectively. Let $(R, +, \cdot)$ be a Vinberg ring and $\partial$ a derivation of $R$. Let $F[x_1, \ldots, x_{m+s}]$ be the polynomial ring on the variables $x_1, \ldots, x_{m+s}$. Let $g_1, \ldots, g_n$ be given polynomials in $F[x_1, \ldots, x_{m+s}]$. For $n, m, s \in N$, we define the $F$ - algebra $F_{n,m,s} = F[e^{a_1}, \ldots, e^{a_n}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}, x_{m+1}, \ldots, x_{m+s}]$ with the standard basis $[3]$

$$B = \{e^{a_1}g_1 \ldots e^{a_n}g_n x_1^{i_1} \ldots x_m^{i_m} \cdot x_{m+1}^{i_{m+1}} \ldots x_{m+s}^{i_{m+s}} | a_1, \ldots, a_n, i_1, \ldots, i_{m+s} \in Z, i_{m+1}, \ldots, i_{m+s} \in N\}$$

(1.2)

and with the obvious addition and the multiplication $[3, 4, 6, 7]$. We define the $F$-Vector space $VN_{n,m,s}$ with the standard basis

$$\{e^{a_1}g_1 \ldots e^{a_n}g_n x_1^{i_1} \ldots x_m^{i_m} \cdot x_{m+1}^{i_{m+1}} \ldots x_{m+s}^{i_{m+s}} \partial_w | a_1, \ldots, a_n, i_1, \ldots, i_{m+s} \in Z, i_{m+1}, \ldots, i_{m+s} \in N, 1 \leq w \leq m + s\}$$

(1.3)

where $\partial_w$ is the usual partial derivative with respect to $x_w$. We define the multiplication $\ast$ on $VN_{n,m,s}$ as

$$f \partial_w \ast h \partial_u = f \partial_w(h) \partial_u$$

(1.4)

for $f \partial_w$ and $h \partial_u \in VN_{n,m,s}$. Thus we can define the Vinberg-type nonassociative ring $\overline{VN_{n,m,s}}$ with the multiplication in (1.4) and with the set $VN_{n,m,s}$. The nonassociative ring $\overline{VN_{n,m,s}}(s \geq 2)$ is not a Vinberg ring as it does not satisfy (1.1). But $\overline{VN_{0,0,1}}$ is a Vinberg ring. For any element $l = e^{a_1}g_1 \ldots e^{a_n}g_n x_1^{l_1} \ldots x_{m+s}^{l_{m+s}} \partial_w$ $(1 \leq w \leq m + s)$, let us call $l_1, \ldots, l_{m+s}$ the powers of $l$. An ideal in a nonassociative ring is a two sided ideal of it. In this paper, we prove that the ring $\overline{VN_{n,m,s}}$ is simple. The ring $\overline{VN_{n,m,s}}$ is not a Jordan ring. The right annihilators of $\overline{VN_{n,m,s}}$ is the sub ring $T_s = \{\sum_{t=1}^{s} c_t d_t | c_t \in F\}$, and the right annihilator of $\overline{VN_{n,m,s}}$ is $[0]$. We can see that the center of $\overline{VN_{n,m,s}}$ is $[0]$ since for any $l \in \overline{VN_{n,m,s}}$, there is $l_1 \in \overline{VN_{n,m,s}}$ such that $[l, l_1] = l_1 l - l - l_1 \neq 0$. In $\overline{VN_{n,m,s}}, \{x_l \partial_t + c_t \partial_l | 1 \leq t \leq m+s, c_t \in F\}$ is a set of orthogonal idempotents in $\overline{VN_{n,m,s}},$ and $\{\sum_{v=1}^{n+s} x_v c_v \partial_v, c_v \in F\}$ is the set of right units of $\overline{VN_{n,m,s}},$ where $n \leq m + s$. A nonassociative ring $VN$ is called power-associative if the subring $F[a]$ generated by any element $a$ of $VN$ is associative (see [8]). From $(a^n \partial + a^n \partial) \ast a^n \partial = a^n \partial \ast (a^n \partial + a^n \partial)$, we know that the algebra $\overline{VN_{n,m,s}}$ is not power associative.

2. Main results

Theorem 2.1. The algebra $\overline{VN_{n,m,s}}$ is simple.
Proof: First we show that the ideal \((\partial_w)\) generated by \(\partial_w\), where \(1 \leq w \leq m + s\), is \(VN_{n,m,s}\). For any basis element \(e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u\) of \(VN_{n,m,s}\) with \(a_k \neq 0\), we have  
\[\partial_k \ast \frac{1}{a_k}e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u = e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u \]  
and similarly the left annihilator is the set 
\[\langle \partial_w \rangle \]  
for \(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n, i_1, \ldots, i_m \in \mathbb{Z}\) and \(i_{m+1}, \ldots, i_{m+s} \in N\), where \(x_k^{i_k}\) means that the term \(x_k^{i_k}\) is omitted. For any 
\[e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u \]  
we have \(x_k^{i_k}\partial_k \ast \frac{1}{a_k}e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u = e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u \). This implies that 
\[e^{a_1g_1} \cdots e^{a_ng_n}x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}l^{i_m+1} \cdots x_{m+s}^{i_{m+s}}\partial_u \]  
holds for any \(k \in Z\) or \(i_k \in N\). Therefore, we have proved that \(\langle \partial_w \rangle = VN_{n,m,s}\). Let \(I\) be a non-zero ideal of \(VN_{n,m,s}\). Let us prove the theorem by induction on the number of distinct homogeneous components of any non-zero element \(l\) in \(I\). Assume that \(l\) has only one \((0, \ldots, 0)\) homogeneous component. We may assume that \(l\) has positive powers from \(l_2 = l_1 \ast l \in I\) by taking an appropriate element \(l_1 \in VN_{n,m,s}\). We can get the element 
\[\partial_1 \ast \cdots \partial_q \ast (\cdots (\partial_q \ast (\cdots (\partial_1 \ast (l_1 \ast l) \cdots) = c\partial_{q_1}\]  
(2.1)
by taking appropriate \(q_1, \ldots, q_t, 1 \leq q_1, \ldots, q_t \leq m + s\), and applying \(\partial_1, \ldots, \partial_q\), in (2.1) with appropriate times, where \(c\) is a non-zero scalar. This implies that 
\(VN_{n,m,s} = \langle \partial_w \rangle \subset I\). Therefore, we have the theorem. Assume that \(l\) is in the \((a_1, \ldots, a_n)\) homogeneous component, then 
\[0 \neq e^{a_1g_1} \cdots e^{a_ng_n}\partial_k \ast l \in VN_{n,0,0}\]  
by taking an appropriate \(t, 1 \leq t \leq m + s\), where at least one of \(a_1, \ldots, a_n\) is not zero. In this case, we have the theorem already. We may assume that \(l\) has \(n\) homogeneous components by induction. Let us assume that \(l\) has the \((0, \ldots, 0, a_{w_1}, \ldots, a_{w_n})\) homogeneous component such that \(a_{w_k} \neq 0\). By taking \(l_1 = e^{-a_{w_1}g_1} \cdots e^{-a_{w_n}g_n}x_1^{i_1}x_2^{i_2} \cdots x_{m+s}^{i_{m+s}}\partial_u\), where \(i_1, \ldots, i_{m+s}\) are sufficiently large positive integers so that \(l_1 \ast l \in I\) has positive powers. By taking an appropriate \(\partial_k, 1 \leq k \leq m + s\), we have \(0 \neq \partial_k \ast (\cdots (\partial_k \ast (l_1 \ast l) \cdots) \in I\) with appropriate times so that 
\[\partial_k \ast (\cdots (\partial_k \ast (l_1 \ast l) \cdots) \neq 0\]  
has almost \(n - 1\) homogeneous components. Therefore, we have the theorem by induction.

3. Derivations of \(VN_{0,0,1}\)

The right annihilator of \(l\) in \(VN_{n,m,s}\) is the set \(\{l_1 \in VN_{n,m,s} | l_1 \ast l = 0\}\) and similarly the left annihilator is the set \(\{l_2 \in VN_{n,m,s} | l_2 \ast l = 0\}\). An additive \(F\) linear map \(D\) of \(VN_{n,m,s}\) is a derivation if \(D(l_1 \ast l_2) = D(l_1) \ast l_2 + l_1 \ast D(l_2)\) holds for any \(l_1, l_2 \in VN_{n,m,s}[1]\).

Remark 3.1. Let \(c \in F\). The map \(D_1\) such that \(D_1(cx^i\partial) = cix^{i-1}\partial\) for any basis element \(x^i\partial\) can be extended linearly on \(VN_{0,0,1}\), which is a derivation of \(VN_{0,0,1}\). Similarly, the \(F\) linear map \(D_2\) on \(VN_{0,0,1}\) such that \(D_2(x^i\partial) = (1 - i)x^i\partial\) for any basis element \(x^i\partial\) of \(VN_{0,0,1}\) is a derivation of \(VN_{0,0,1}\).
Lemma 3.2. The left annihilator of $\partial$ is $\overline{N_{0,0.1}}$, and the right annihilator of $\partial$ is $\{c\partial|c \in F\}$.

Proof: The proof is straightforward by the definitions of the right and left annihilators of $\partial$ in $\overline{N_{0,0.1}}$.

\[ \square \]

Theorem 3.3. For any derivation $D$ of $\overline{N_{0,0.1}}$, $D = c_1D_1 + c_2D_2, c_1, c_2 \in F$, where $D_1$ and $D_2$ are the derivations of $\overline{N_{0,0.1}}$ in Remark 3.1.

Proof: Let $D$ be any derivation of $\overline{N_{0,0.1}}$. Then

\[ D(\partial * \partial) = D(\partial) * \partial + \partial * D(\partial) = \partial * D(\partial) = 0. \]

By Lemma 3.1, we have

\[ D(\partial) = C(0)\partial \text{ for some } C(0) \in F. \quad (3.1) \]

By $D(\partial * x\partial) = D(\partial) * x\partial + \partial * D(x\partial) = C(0)\partial = C(0)\partial + \partial * D(x\partial)$, we have

\[ D(x\partial) = C(1)\partial \text{ for some } C(1) \in F. \quad (3.2) \]

This implies that $D(\partial * x^2\partial) = 2D(x\partial) = 2C(1)\partial$. But,

\[ D(\partial) * x^2\partial + \partial * D(x^2\partial) = C(0)\partial * x^2\partial + \partial * D(x^2\partial) = 2C(0)x\partial + \partial * D(x^2\partial). \]

This implies that $\partial * D(x^2\partial) = -2C(0)x\partial + 2C(1)\partial$. Then $D(x^2\partial) = -C(0)x^2\partial + 2C(1)x\partial + C(2,0)\partial$ for some $C(2,0) \in F$. We have

\[ D(x\partial * x^2\partial) = 2D(x^2\partial) = -2C(0)x^2\partial + 4C(1)x\partial + C(2,0). \quad (3.3) \]

Also, we have

\[ D(x\partial) * x^2\partial + x\partial * D(x^2\partial) = 2C(1)x\partial + x\partial * (-C(0)x^2\partial) + 2C(1)x\partial + C(2,0)\partial. \]

Thus

\[ D(x\partial) * x^2\partial + x\partial * D(x^2\partial) = -2C(0)x^2\partial + 4C(1)x\partial. \quad (3.4) \]

By (3.3) and (3.4), we have $C(2,0) = 0$. Let us assume that

\[ D(x^n\partial) = C(0)(1-n)x^n\partial + C(1)n x^{n-1}\partial \text{ for some fixed } n \in N, \text{ by induction.} \]

Thus we have

\[ D(\partial * x^{n+1}\partial) = (n+1)D(x^n\partial) = (n+1)C(0)(1-n)x^n\partial + (n+1)C(1)n x^{n-1}\partial. \]

But we have $D(\partial) * x^{n+1}\partial + \partial * D(x^{n+1}\partial) = C(0)(n+1)x^n\partial + \partial * D(x^{n+1}\partial)$. This implies that

\[ \partial * D(x^{n+1}\partial) = -C(0)(n+1)x^n\partial + C(0)(n+1)(1-n)x^n\partial + C(1)n(n+1)x^{n-1}\partial = -nC(0)(n+1)x^n\partial + C(1)n(n+1)x^{n-1}\partial. \]

Hence,

\[ D(x^{n+1}\partial) = -nC(0)x^{n+1}\partial + C(1)(n+1)x^n\partial + C(n,0)\partial, C(n,0) \in F. \]
Then
\[ D(x\partial \ast x^{n+1}\partial) = (n+1)D(x^{n+1}\partial) = -nC(0)(n+1)x^{n+1}\partial + C(1)(n+1)^2x^n\partial + C(n,0)(n+1)\partial. \]

On the other hand, we have
\[ C(1)\partial \ast x^{n+1}\partial + x\partial \ast (-nC(0)x^{n+1}\partial + C(1)(n+1)x^n\partial + C(n,0)\partial) = -nC(0)(n+1)x^{n+1}\partial + C(1)(n+1)^2x^n\partial. \]

This implies that \( C(n,0) = 0 \). Therefore, we have proved that
\[ D(x^n\partial) = C(0)\partial + C(1)x^n\partial, n \in \mathbb{N}. \]
This shows that \( D = C(0)D_2 + C(1)D_1 \) and completes the proof of the theorem.

\[ \square \]

4. Solid Algebras

Let \( A \) be an \( F \)-algebra. Let \( \text{End}_F(A) \) be the set of all \( F \)-endomorphisms of \( A \), and \( \text{Aut}_F(A) \) the set of all automorphisms of \( A \). An \( F \)-algebra \( A \) is solid if every non-zero endomorphism of \( A \) is surjective.

**Proposition 4.1.** A simple algebra \( A \) is solid if and only if \( \text{End}_F(A) = \{0\} \cup \text{Aut}_F(A) \).

**Proof:** It is straightforward by the fact that \( A \) is a simple algebra and the definition of the solid algebra. \( \square \)

**Lemma 4.2.** For any \( \theta \in \text{End}_F(V_{N,0,1}) \), if \( \theta(\partial) = 0 \), then \( \theta \) is the zero map of \( V_{N,0,1} \).

**Proof:** We have \( \theta(\partial \ast x^n\partial) = n\theta(x^{n-1}\partial) = 0 \) for any \( n \in \mathbb{N} \), which implies that \( \theta \) is the zero map by induction on the degree of \( x^n\partial \). \( \square \)

**Lemma 4.3.** For any non-zero \( F \)-endomorphism \( \theta \) of \( V_{N,0,1} \), \( \theta(\partial) = c_0\partial \) holds for some fixed \( 0 \neq c_0 \in F \).

**Proof:** We have \( \theta(\partial \ast \partial) = \theta(\partial) \ast \theta(\partial) = 0. \) Since \( \theta(\partial) \neq 0 \), by Lemma 4.1, we have \( \theta(\partial) = c_0\partial, 0 \neq c_0 \in F. \) \( \square \)

**Proposition 4.4.** If \( \theta \) is a non-zero endomorphism of \( V_{N,0,1} \), then \( \theta \) is an epimorphism.

**Proof:** By Lemma 4.3 we have \( \theta(\partial) = c_0\partial \) for some non-zero \( c_0 \in F \). From
\[ \theta(\partial \ast x\partial) = \theta(\partial), \]
we have $c_0 \partial \ast \theta(x \partial) = c_0 \partial$. This implies that $\theta(x \partial) = c_1 \partial + x \partial$ for some $c_1 \in F$. By $\theta(\partial \ast x^2 \partial) = 2\theta(x \partial)$, we have $\theta(x^2 \partial) = c_r \partial + \frac{2c_r x}{c_0} \partial + \frac{x_0^2}{c_0} \partial$ for $c_r \in F$. By $\theta(x \partial \ast x^2 \partial) = 2\theta(x^2 \partial)$, we have

$$
(c_1 \partial + x \partial) \ast (c_r \partial + \frac{2c_r x}{c_0} \partial + \frac{x^2}{c_0} \partial) = 2c_r \partial + \frac{4c_1 x}{c_0} \partial + \frac{2x^2}{c_0} \partial. \quad (4.1)
$$

By comparing the coefficients of both sides of (4.1), we have $c_r = \frac{c_1^2}{c_0}$. Thus, we have $\theta(x^2 \partial) = c_0^{-1}(x + c_1)^2 \partial$. Let us assume that $\theta(x^n \partial) = c_0^{-n}(x + c_1)^n \partial$ for some fixed non-negative integer $n$ inductively.

From $\theta(\partial \ast x^{n+1} \partial) = (n+1)\theta(x^n \partial)$, we have

$$
\partial \ast \theta(x^{n+1} \partial) = (n+1)c_0^{-n}(x + c_1)^n \partial.
$$

This implies that $\theta(x^{n+1} \partial) = c_0^{-n}(x + c_1)^{n+1} \partial + c_n \partial$ for some $c_n \in F$. By

$$
\theta(x \partial \ast x^{n+1} \partial) = (n+1)\theta(x^{n+1} \partial), \quad (4.2)
$$

we have $(x + c_1) \partial \ast (c_0^{-n}(x + c_1)^{n+1} \partial + c_n \partial) = c_0^{-n}(n+1)(x + c_1)^{n+1} \partial + (n+1)c_n \partial$. By comparing the coefficients of both sides of (4.2), we have $c_n = 0$. Thus, $\theta(x^n \partial) = c_0^{-n}(x + c_1)^m \partial$ holds for any $m \in F$ inductively. Therefore, any $l \in V_{N_{0,0,1}}$ can be written as

$$
l = c_0^{r_0-1}(x + c_1)^{r_1} \partial + \cdots + c_0^{r_0-1} \partial = c_n^{r} \theta(x^d \partial) + \cdots + c_0^{r_0} \theta(\partial),
$$

Where $c_1, \ldots, c_0 \in F$. This implies that $\theta$ is surjective. The following corollary is the version of Jacobian conjecture on $V_{N_{0,0,1}}$.

\textbf{Corollary 4.5.} For any non-zero endomorphism $\theta$ of $V_{N_{0,0,1}}$, $\theta$ is an automorphism of $V_{N_{0,0,1}}$.

\textbf{Proof:} By Lemma 4.3, $\theta(\partial) = c_0 \partial$ for some non-zero $c_0 \in F$. Since $V_{N_{0,0,1}}$ is simple, $\theta$ is one to one. By Proposition 4.4, $\theta$ is onto. \hfill $\Box$

\textbf{Corollary 4.6.} $\text{End}(V_{N_{0,0,1}}) = \text{Aut}(V_{N_{0,0,1}}) \cup \{0\}$, where 0 is the zero map of $V_{N_{0,0,1}}$.

\textbf{Proof:} It is straightforward by Corollary 4.5. \hfill $\Box$

By Corollary 4.6, we know that $V_{N_{0,0,1}}$ is solid.

\textbf{Proposition 4.7.} For any $\theta \in \text{Aut}(V_{N_{n,m,s}})$, we have $\theta(T_{s_1}) = T_{s_1}$.

\textbf{Proof:} Since $T_{s_1}$ is the unique maximal right annihilator of $V_{N_{n,m,s}}$, $\theta(T_{s_1}) = T_{s_1}$ holds for any $\theta \in \text{Aut}(V_{N_{n,m,s}})$. \hfill $\Box$

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