On the Existence Results for a Class of Singular Elliptic System Involving Indefinite Weight Functions and Asymptotically Linear Growth Forcing Terms

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ABSTRACT: In this work, we study the existence of positive solutions to the singular system

\[
\begin{align*}
-\Delta_p u &= \lambda a(x) f(v) - u^{-\alpha} \quad \text{in } \Omega, \\
-\Delta_p v &= \lambda b(x) g(u) - v^{-\alpha} \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \lambda \) is a positive parameter, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u), \) \( p > 1, \) \( \Omega \subset \mathbb{R}^n \) some for \( n > 1, \) is a bounded domain with smooth boundary \( \partial \Omega, \) \( 0 < \alpha < 1, \) and \( f, g : [0, \infty) \to \mathbb{R} \) are continuous, nondecreasing functions which are asymptotically \( p \)-linear at \( \infty. \) We prove the existence of a positive solution for a certain range of \( \lambda \) using the method of sub-supersolutions.

Key Words: Infinite semipositone problems; Indefinite weight; Asymptotically linear growth forcing terms; Sub-supersolution method.

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1. Introduction

In this article, we mainly consider the existence of a positive solution of the following singular elliptic system

\[
\begin{align*}
-\Delta_p u &= \lambda a(x) f(v) - u^{-\alpha} \quad \text{in } \Omega, \\
-\Delta_p v &= \lambda b(x) g(u) - v^{-\alpha} \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \lambda \) is a positive parameter, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u), \) \( p > 1, \) \( \Omega \subset \mathbb{R}^n \) some for \( n > 1, \) is a bounded domain with smooth boundary \( \partial \Omega, \) \( 0 < \alpha < 1, \) and \( f, g : [0, \infty] \to \mathbb{R} \) are continuous, nondecreasing functions which are asymptotically \( p \)-linear at \( \infty. \) We prove the existence of a positive solution for a certain range of \( \lambda. \)

We consider problem (1.1) under the following assumptions.

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(H_1) There exist \( \sigma_1 > 0, k_1 > 0 \) and \( s_1 > 1 \) such that
\[
f(s) \geq \sigma_1 s^{p-1} - k_1
\]
for every \( s \in [0, s_1] \)

and that there exist \( \sigma_2 > 0, k_2 > 0 \) and \( s_2 > 1 \) such that
\[
g(s) \geq \sigma_2 s^{p-1} - k_2
\]
for every \( s \in [0, s_2] \),

(H_2) For all \( M > 0 \), \( \lim_{s \to +\infty} \frac{f(Ms)}{s^{p-1}} = \sigma \) for some \( \sigma > 0 \).

(H_3) \( a, b : \overline{\Omega} \to (0, \infty) \) are continuous functions such that \( a_1 = \min_{x \in \overline{\Omega}} a(x) \), \( b_1 = \min_{x \in \overline{\Omega}} b(x) \), \( a_2 = \max_{x \in \overline{\Omega}} a(x) \) and \( b_2 = \max_{x \in \overline{\Omega}} b(x) \).

(H_4) There exists \( \tau \in \mathbb{R} \) such that for each \( M > 0 \), \( f(Ms) \leq M^\tau f(s) \) for \( s \gg 1 \).

Let \( F(u) := \lambda a(x)f(u) - u^{-\alpha} \). The case when \( F(0) < 0 \) (and finite) is referred to in the literature as a semipositone problem. Finding a positive solution for a semipositone problem is well known to be challenging (see [2,5]). Here we consider the more challenging case when \( \lim_{u \to 0^+} F(u) = -\infty \), which has received attention very recently and is referred to as an infinite semipositone problem. However, most of these studies have concentrated on the case when the nonlinear function satisfies a sublinear condition at \( \infty \) (see [6,7,9]). We refer to [15,16,17,18,19] for additional results on elliptic problems. The only paper to our knowledge dealing with an infinite semipositone problem with an asymptotically linear nonlinearity is [8], where the author is restricted to the case \( p = 2 \). Also here the existence of a positive solution is focused near \( \frac{\lambda_1}{\mu^\alpha} \), where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\).

See also [1,11], where asymptotically linear nonlinearities have been discussed in the case of a nonsingular semipositone problem and an infinite positone problem. Motivated by the above papers, in this note, we are interested in the existence of positive solution for problem (1.1), where \( a, b \) are continuous functions in \( \overline{\Omega} \) and \( \lambda \) is a positive parameter. Our main goal is to improve the result introduced in [12], in which the authors study the existence of positive solutions for an infinite semipositone problem with the nonlinearity \( f \) being not dependent of \( x \). We shall establish our an existence result via the method of sub and supersolutions.

**Definition 1.1.** We say that \( (\psi_1, \psi_2) \) (resp. \( (z_1, z_2) \)) in \( (W^{1,p}(\Omega) \cap C(\overline{\Omega}), W^{1,p}(\Omega) \cap C(\overline{\Omega})) \) are called a subsolution (resp. a supersolution) of (1.1), if \( \psi_i \) (\( i = 1, 2 \)) satisfy
\[
\begin{cases}
\int_\Omega |\nabla \psi_1(x)|^{p-2} \nabla \psi_1 \cdot \nabla w_1 dx \leq \int_\Omega \left( \lambda a(x)f(\psi_2) - \psi_1^{-\alpha} \right) w_1(x) dx \\
\int_\Omega |\nabla \psi_2(x)|^{p-2} \nabla \psi_2 \cdot \nabla w_2 dx \leq \int_\Omega \left( \lambda b(x)g(\psi_1) - \psi_2^{-\alpha} \right) w_2(x) dx \\
\psi_1, \psi_2 > 0 \\
\psi_1 = \psi_2 = 0
\end{cases}
\] in \( \Omega \),
on \( \partial \Omega \),
\[(1.2)\]
1.1 Condition. By anti-maximum principle (see [132]), for all \( x \) positive constants \( s \) Theorem 2.1.

The following lemma was established by Miyagaki in [14]:

**Lemma 1.1** (See [14]). *If there exist sub-supersolutions \((\psi_1, \psi_2)\) and \((z_1, z_2)\), respectively, such that \( 0 \leq \psi_i(x) \leq z_i(x) \) \((i = 1, 2)\) for all \( x \in \Omega \), then (1.1) has a positive solution \((u, v)\) such that \( \psi_1(x) \leq u(x) \leq z_1(x) \) and \( \psi_2(x) \leq v(x) \leq z_2(x) \) for all \( x \in \Omega \).*

2. Main result

With the hypotheses introduced in previous section, the main result of this paper is given by the following theorem.

**Theorem 2.1.** *Assume the conditions \((H_1) - (H_4)\) are satisfied. Then there exist positive constants \( s_0^* (\sigma, \Omega), J^* (\Omega) \), and \( \lambda_e, \lambda_+ \) such that if \( \min \{ s_1, s_2 \} \geq s_0^* \) and \( \min(\sigma_1, \sigma_2) \lambda_e \geq J^* \), problem (1.1) has a positive solution for \( \lambda \in [\lambda_e, \lambda_+] \).*

**Proof:** Let \( \mu_1 \) is the principal eigenvalue of operator \(-\Delta_p\) with Dirichlet boundary condition. By anti-maximum principle (see [10]), there exists \( \xi = \xi(\Omega) > 0 \) such that the solution \( z_\mu \) of

\[
\begin{cases}
-\Delta_p z - \mu |z|^{p-2} z = -1 & \text{in } \Omega, \\
z = 0 & \text{on } \partial \Omega,
\end{cases}
\]

for \( \mu \in (\mu_1, \mu_1 + \xi) \) is positive in \( \Omega \) and is such that \( \frac{\partial z_\mu}{\partial \nu} < 0 \) on \( \partial \Omega \), where \( \nu \) is outward normal vector at \( \partial \Omega \).

Since \( z_\mu > 0 \) in \( \Omega \) and \( \frac{\partial z_\mu}{\partial \nu} < 0 \) there exist \( m > 0 \), \( A > 0 \), and \( \delta > 0 \) be such that \( |\nabla z_\mu| \geq m \) in \( \Omega_\delta \) and \( z_\mu \geq A \) in \( \Omega \setminus \Omega_\delta \), where \( \Omega_\delta = \{ x \in \Omega : d(x, \partial \Omega) \leq \delta \} \).

We first construct a supersolution for (1.1). Let

\[
(z_1, z_2) = \left( M_\lambda e_p, |\lambda b_2 g(M_\lambda \|e_p\|_\infty)|^{-1} e_p \right),
\]
where \( M_\lambda \gg 1 \) is a large positive constant and \( e_p \) is the unique positive solution of
\[
\begin{align*}
-\Delta_p e_p &= 1 \quad \text{in } \Omega, \\
e_p &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
By the hypothesis \((H_2)\), we can choose \( M_\lambda \gg 1 \) such that
\[
\frac{2\sigma}{a_2} \geq \frac{f\left(\left\|b_2 g(M_\lambda \|e_p\|\infty)\right\|^{\frac{1}{p-1}}\right)}{\left(M_\lambda \|e_p\|\infty\right)^{p-1}}.
\]
Then
\[
-\Delta_p z_1 = M_\lambda^{p-1} \geq \frac{a_2 \left(\left\|b_2 g(M_\lambda \|e_p\|\infty)\right\|^{\frac{1}{p-1}}\right)}{2\sigma \|e_p\|_{\infty}^{p-1}}.
\]
Now since \( \lambda \leq \frac{1}{\|e_p\|_{\|e_p\|\infty}^{p-1} (2\sigma)^{p-1+\sigma}} = \lambda_{**} \), we have
\[
-\Delta_p z_1 \geq \frac{\left(\lambda^{\frac{p-1+\sigma}{p}} a_2 \|e_p\|_{\|e_p\|\infty}^{p-1+\sigma} f\left(\left\|b_2 g(M_\lambda \|e_p\|\infty)\right\|^{\frac{1}{p-1}}\right)\right)}{\|e_p\|_{\infty}^{p-1}}
\geq \frac{\lambda a_2 \lambda^{\frac{1}{p-\sigma}} \|e_p\|_{\|e_p\|\infty}^{\frac{p+1}{p-\sigma}} f\left(\left\|b_2 g(M_\lambda \|e_p\|\infty)\right\|^{\frac{1}{p-1}}\right)\}}{\|e_p\|_{\infty}^{p-1}}.
\]
Note that \((H_2)\) implies \( g(s) \rightarrow \infty \) as \( s \rightarrow \infty \). Hence from \((H_3)\) for \( M_\lambda \gg 1 \) we get
\[
-\Delta_p z_1 \geq \lambda a_2 \frac{f\left(\|e_p\|_{\|e_p\|\infty} \left[\lambda b_2 g(M_\lambda \|e_p\|\infty)\right]^{\frac{1}{p-1}}\right)}{\|e_p\|_{\|e_p\|\infty}^{p-1}}
\geq \lambda a_2 f(z_2) - \frac{1}{z_1^\alpha}
\geq \lambda a(x) f(z_2) - \frac{1}{z_1^\alpha}.
\]
Also
\[
-\Delta_p z_2 = \lambda b_2 g(M_\lambda \|e_p\|\infty) \geq \lambda b_2 g(M_\lambda e_p) \geq \lambda b_2 g(z_1) - \frac{1}{z_2^\sigma} \geq \lambda b(x) g(z_1) - \frac{1}{z_2^\sigma}.
\]
Hence, from relations \((2.1)\) and \((2.2)\) we see that \((z_1, z_2)\) is a supersolution of problem \((1.1)\) when \( \lambda \leq \frac{1}{\|e_p\|_{\|e_p\|\infty}^{p-1} (2\sigma)^{p-1+\sigma}}. \)

Define
\[
(\psi_1, \psi_2) := \left(k_0 z_\mu^\frac{-\alpha}{p-1+\alpha}, k_0 z_\mu^\frac{-\alpha}{p-1+\alpha}\right)
\]
where \( k_0 \gg 0 \) is such that
\[
\frac{1}{k_0^{p-1+\alpha}} \left(1 + \frac{kk_0^\frac{-\alpha}{p-1+\alpha}}{\|e_p\|_{\|e_p\|\infty}^{p-1} (2\sigma)^{p-1+\sigma}}\right) \leq \min \{\{x_1, x_2\}\}.
\]
with $k = \max\{k_1, k_2\}$ and $(x_1, x_2) = \left(\frac{p^p (1 - \alpha (p - 1))^{p - 1}}{(p - 1 + \alpha)^p}, \frac{p}{p - 1 + \alpha}\right)$. Then
\[\nabla \psi_1 = k_0 \left(\frac{p}{p - 1 + \alpha}\right) z_{\mu}^{1 - \alpha} \nabla z_{\mu}\]
and
\[-\Delta p \psi_1
= -\text{div}(\nabla |\nabla \psi_1|^{p - 2} \nabla \psi_1)
= -k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} \text{div}\left(z_{\mu}^{1 - \alpha (p - 1)} |\nabla z_{\mu}|^{p - 2} \nabla z_{\mu}\right)
= -k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} \left\{\nabla |\nabla z_{\mu}|^{p - 2} \nabla z_{\mu} + z_{\mu}^{(1 - \alpha (p - 1))} \Delta z_{\mu}\right\}
= -k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} \left\{(1 - \alpha) \left(\frac{p - 1}{p - 1 + \alpha}\right) z_{\mu}^{1 - \alpha (p - 1)} |\nabla z_{\mu}| + z_{\mu}^{(1 - \alpha (p - 1))} (1 - \mu z_{\mu}^{-1})\right\}
= k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} \mu z_{\mu}^{1 - \alpha (p - 1)} - k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} z_{\mu}^{1 - \alpha (p - 1)}
= k_0^{-1} p^{p - 1} (1 - \alpha) (p - 1) |\nabla z_{\mu}|^p.
\]
(2.4)

Now we let $s_0^p(\sigma, \Omega) = k_0 \|z_{\mu}^{1 - \alpha}\|_{\infty}$. If we can prove
\[-\Delta p \psi_1 \leq \lambda_1 \sigma_1 k_0^{-1} z_{\mu}^{\frac{p (p - 1)}{p - 1 + \alpha}} - \lambda k - \frac{1}{k_0} z_{\mu}^{1 - \alpha (p - 1)},
\]
(2.5)
then it implies from (H1) that
\[-\Delta \psi_1 \leq \lambda_1 f(\psi_2) - \frac{1}{\psi_1} \leq \lambda a(x) f(\psi_2) - \frac{1}{\psi_1}.
\]
Let us prove (2.5) holds true. Let $\lambda_1 = \frac{\mu}{\min\{1, \mu\}} \min\{1, \sigma_1, \sigma_2\}$. For $\lambda \geq \lambda_1$, we get
\[k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} \mu z_{\mu}^{\frac{p (p - 1)}{p - 1 + \alpha}} \leq \lambda_1 \sigma_1 k_0^{-1} z_{\mu}^{\frac{p (p - 1)}{p - 1 + \alpha}},
\]
(2.6)
\[k_0^{-1} \left(\frac{p}{p - 1 + \alpha}\right)^{p - 1} \mu z_{\mu}^{\frac{p (p - 1)}{p - 1 + \alpha}} \leq \lambda_1 \sigma_2 k_0^{-1} z_{\mu}^{\frac{p (p - 1)}{p - 1 + \alpha}}.
\]
(2.7)
Also since $\lambda \leq \lambda_1$, we get
\[\lambda k + \frac{1}{k_0} \left[\frac{p (p - 1)}{p - 1 + \alpha}\right] \leq \frac{1}{k_0} \left[\frac{p (p - 1)}{p - 1 + \alpha}\right] + \frac{k}{\|e_p\|_{L^\infty}}
= \frac{k_0^{-1}}{z_{\mu}} \left[1 + \frac{kk_0^{-1} \sigma_1}{\|e_p\|_{L^\infty}^1 (2\sigma)^{p - 1 + \alpha}}\right],
\]
(2.8)
Now in $\Omega_\delta$, we have $|\nabla z_\mu| \geq m$, and by (2.3)
\[
\frac{1}{k_0^{p-1+\alpha}} \left( 1 + \frac{kk_0^p z_\mu^{\frac{p-1+\alpha}{p}}}{\|e_\mu\|_\infty^{p-1} (2\sigma)^{\frac{p-1+\alpha}{p}}} \right) \leq \frac{m^p(1-\alpha)(p-1)p^{p-1}}{(p-1+\alpha)^p}.
\]

Hence,
\[
\lambda k_0^{p-1+\alpha} \leq \frac{k_0^{p-1} p^{p-1}}{(p-1+\alpha)^p} (1-\alpha)(p-1)p^{p-1} \leq \frac{k_0^{p-1} p^{p-1}}{(p-1+\alpha)^p} \parallel e_\mu \parallel_\infty^{p-1} \left( 1 + \frac{kk_0^p z_\mu^{\frac{p-1+\alpha}{p}}}{\|e_\mu\|_\infty^{p-1} (2\sigma)^{\frac{p-1+\alpha}{p}}} \right).
\]
(2.9)

From (2.6) and (2.9) it can be seen that (2.5) holds in $\Omega_\delta$. A similar argument shows that
\[
-\Delta_p \psi_2 \leq \lambda_1 g(\psi_1) - \frac{1}{\psi_2^p} \leq \lambda_1 g(\psi_1) - \frac{1}{\psi_2^p}.
\]
We will now prove (2.5) holds also in $\Omega \setminus \Omega_\delta$. Since $z_\mu \geq A$ in $\Omega \setminus \Omega_\delta$ and by (2.3) and (2.8) we get
\[
\lambda k_0^{p-1+\alpha} \leq \frac{k_0^{p-1} p^{p-1}}{(p-1+\alpha)^p} (1-\alpha)(p-1)p^{p-1} \leq \frac{k_0^{p-1} p^{p-1}}{(p-1+\alpha)^p} \parallel e_\mu \parallel_\infty^{p-1} \left( 1 + \frac{kk_0^p z_\mu^{\frac{p-1+\alpha}{p}}}{\|e_\mu\|_\infty^{p-1} (2\sigma)^{\frac{p-1+\alpha}{p}}} \right).
\]
(2.10)

From (2.6) and (2.10), (2.5) holds also in $\Omega \setminus \Omega_\delta$.

Thus $(\psi_1, \psi_2)$ is a positive subsolution of (1.1) if $\lambda \in [\lambda_*, \lambda_{**}]$. We can now choose $M_\lambda \gg 1$ such that $\psi_1 \leq z_1, \psi_2 \leq z_2$. Let
\[
J^*(\Omega) = 2^{\frac{p-1+\alpha}{p}} \|e_\mu\|_\infty^{p-1} \left( \frac{p}{p-1+\alpha} \right)^{p-1} \lambda_1 g(\psi_1) \left( \frac{p}{p-1+\alpha} \right)^{p-1}.
\]
If $\frac{\min(a_1, b_1) \min(\sigma_1, \sigma_2)}{(2\sigma)^{\frac{p-1+\alpha}{p}}} \geq J^*$ it is easy to see that $\lambda_* \leq \lambda_{**}$ and for $\lambda \in [\lambda_*, \lambda_{**}]$ we have a positive solution. This completes the proof of Theorem 2.1.

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