Reidemeister Classes for Coincidences Between Sections of a Fiber Bundle

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ABSTRACT: Let \( s_0, f_0 \) be two sections of a fiber bundle \( q : E \to B \) and assume the coincidence set \( \Gamma(s_0, f_0) \neq \emptyset \). We consider the problem of identifying the algebraic Reidemeister classes for \( s_0 \) and \( f_0 \) with the geometric classes obtained by the lifting maps on covering spaces. We do this by using the homotopy lifting extension property of the fibration \( q \) to obtain homotopies over \( B \). When we make this and the basic point is fixed we can use the elements \( s_0(\beta), f_0(\beta^{-1}) \) where \( \beta \in \pi_1(B, b_0) \) and the elements \( \gamma \in \pi_1(F_0, e_0) \). So we will introduce the algebraic Reidemeister classes relative to the subgroup \( \pi_1(F_0, e_0) \). When the basic points are not fixed we need to consider the classes \( \tilde{s}_0 \) of lifting of \( s_0 \) defined on the universal covering \( \tilde{B} \) to \( \tilde{E} \). The present work relates the lifting classes \( \tilde{s}_0 \) of \( s_0 \) and the algebraic Reidemeister classes \( R_A(s_0, f_0; \pi_1(F_0, e_0)) \), as given in [2],[3] and [5].

Key Words: Coincidence theory, Reidemeister classes, Fiber bundle.

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1. Introduction

Let \( f : (B, b_0) \to (B, b_0) \) be a function and assume that \( B \) is compact, locally path connected, semi locally 1-connected and an euclidean neighborhood retract space. Then there is the universal covering \( p^{b_0} : \tilde{B}(b_0) \to B \), constructed from the trivial subgroup \( \{[b_0]\} \leq \pi_1(B, b_0) \). Let \( T : \pi_1(B, b_0) \to \text{Cov}(\tilde{B}(b_0)/B) \) be the isomorphism \( \beta \mapsto T_\beta \), the deck transformation associated to \( \beta \).

Let \( \mathcal{L}(f_0) \) be the set of all liftings \( f \) of \( f_0 \) with respect to the following commutative diagram. Note that the second diagram corresponds to the case when \( f \) is
the identity map $I_B$.

$$\begin{array}{ccc}
\tilde{B}(b_0) & \xrightarrow{\tilde{f}} & \tilde{B}(b_0) \\
p^0 \downarrow & & \downarrow p^0 \\
B & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc}
\tilde{B}(b_0) & \xrightarrow{I_B} & \tilde{B}(b_0) \\
p^0 \downarrow & & \downarrow p^0 \\
B & \xrightarrow{I_B} & B
\end{array} \quad (1.1)
$$

Consider the following equivalence relation on the set $\mathcal{L}(f)$: $\tilde{f}_1 R_L \tilde{f}_2 \iff \tilde{f}_1 = T_\beta \circ \tilde{f}_2 \circ T_\beta^{-1}$, with $\beta \in \pi_1(B, b_0)$. We denote by $R_L(\mathcal{L}(f))$ the quotient set and by $[\tilde{f}]_L$ the class of $\tilde{f}$ and $r_L(\mathcal{L}(f)) = |R_L(\mathcal{L}(f))|$. In [3], [2] and [4] the authors related the relation $R_L$ with the algebraic Reidemeister classes induced by $I_B, f : \pi_1(B, b_0) \to \pi_1(B, b_0)$ whose quotient set is $R_A(f, I_B)$ and $r_A(f, I_B) = |R_A(f, I_B)|$. They proved that:

1. There is an one to one correspondence between $R_L(\mathcal{L}(f))$ and $R_A(f, I_B)$, therefore $r_L(\mathcal{L}(f)) = r_A(f, I_B)$.

2. If $[\tilde{f}_1]_L = [\tilde{f}_2]_L$ then $p^0_b \left( Fix(\tilde{f}_1) \right) = p^0_b \left( Fix(\tilde{f}_2) \right)$.

3. If $p^0_b \left( Fix(\tilde{f}_1) \right) \cap p^0_b \left( Fix(\tilde{f}_2) \right) \neq \emptyset$ then $[\tilde{f}_1]_L = [\tilde{f}_2]_L$.

In fixed point theory it is usual to put the date $f, I_B : B \to B$ on the context of fiber bundle considering the trivial fiber bundle $q : B \times B \to B$, $q(b_1, b_2) = b_1$ and the sections $s_0, f_0 : B \to B \times B$ of $q$ given by $s_0(b) = (b, b)$ and $f_0(b) = (b, f(b))$.

In this work we consider a general fiber bundle $q : E \to B$ and initially two sections $s_0, f_0 : (B, b_0) \to (E, e_0)$ and $F_0$ the fiber over $b_0$ which satisfies good hypotheses on $B, E$ and $F_0 = q^{-1}(b_0)$. The purpose of this work is to prove an analogous result of some results in [2] and [3] in this context of section on fiber bundle.

This work is divided in four sections. In Section 2 we established notations and we listed some results about the construction of covering spaces of a subgroup $G$ of $\pi_1(E, e_0)$ and we explicit the lifting $\tilde{s}_0, \tilde{f}_0$ and $s_{F_0}, f_{F_0}$. We also introduce the equivalence relation $R_{F_0}$ on the set $\mathcal{L}(s_0, f_0)$ and the relation $R_{s_0}$ on $\mathcal{L}(f_0, s_{F_0})$ which the sets are, respectively, specified lifting maps on the universal covering for the sections $s_0$ and $f_0$, as in [2],[3] and [5]. The Theorem 2.9 established the first approximations between relations $R_{F_0}$ or $R_{s_0}$ and the Reidemeister relation relative to the subgroup $\pi_1(F_0, e_0)$, as we will see in the next section. In Section 3 we defined the algebraic Reidemeister classes relative of a subgroup $H_0 \leq G_0$ induced by the homomorphisms $\varphi, \psi : G_1 \to G_0$ which the quotient set is $R_A(\varphi, \psi; H_0)$. In particular, we will apply this for the homomorphisms on the fundamental groups for two sections $s_0, f_0 : \pi_1(B, b_0) \to \pi_1(E, e_0)$ of a fiber bundle $q : E \to B$ and the subgroup $H_0 = \pi_1(F_0, e_0)$, so we have the set $R_A(s_0, f_0; \pi_1(F_0, e_0))$.

In Section 4 we also defined the set of Nielsen coincidence classes which is indicated by $\Gamma(s_0, f_0)$ and proved that it is finite under good hypotheses on the spaces
B, E, F₀. We also exhibit an injection map \( \tilde{\Gamma}(s₀, f₀) \to Rₜ₀(s₀, f₀; π₁(F₀, e₀)) \).

We finished this section relating the theorems 2.9 and 4.3 and then we proved the main theorem 4.5, which established an one-to-one correspondence between \( Rₗ₀(\mathcal{L}(f₀; sF₀)) \) and \( Rₜ₀(s₀, f₀; π₁(F₀, e₀)) \) or similarly for the classes \([\pi]₀\) on the set \( Rₗ₀(\mathcal{L}(s₀; fF₀))\), as in [4] and [5].

2. Covering projection constructed from a subgroup and relations on the lifting maps

Let \( G \) be a subgroup of \( π₁(E, e₀) \) and let \( P(E, e₀) \) be the set of all paths \( α : [0, 1] \to E \) such that \( α(0) = e₀ \). We say that \( α₁, α₂ \in P(E, e₀) \) are \( G \)-related if \( α₁(1) = α₂(1) \) and the class \([α₁ * α₂⁻¹]\) \( G \geq π₁(E, e₀) \). It is easy to prove that this is an equivalence relation and we denote the class from the path \( α \) by \((e, α)\) with \( e = α(1) \). The quotient set of \( P(E, e₀) \) by this relation is indicated by \( \tilde{E}(G) \).

In [6] the author defined a basis for a topology on the set \( \tilde{E}(G) \) for which the function \( p^G : \tilde{E}(G) \to E, p^G(e, α) = e \) is continuous. Moreover if \( E \) is path connected then \( p^G \) is a surjection and we have the following statements:

1. If \( E \) is connected, locally path connected and semi-locally 1-connected then
   \[
p^G : \left(\tilde{E}(G), \tilde{e₀}\right) \to (E, e₀) \text{ is a covering space with } p^G(π₁(\tilde{E}(G), \tilde{e₀})) = G,
   \]
   where \( \tilde{e₀} = (e₀, π₀) \) and \( π₀ \) is the constant path on \( e₀ \in E \).

2. If \( G₁ ≤ G₂ ≤ π₁(E, e₀) \) are subgroups and \( p^{G₁} : \tilde{E}(G₁) \to E, p^{G₂} : \tilde{E}(G₂) \to E \) are covering spaces then there is a covering space \( p^{G₁}_G : \tilde{E}(G₁) \to \tilde{E}(G₂) \) so that \( p^{G₁} = p^{G₂} \circ p^{G₁}_G \).

3. If \( G \) is a normal subgroup of \( π₁(E, e₀) \) and \( p : (\tilde{E}, \tilde{x₀}) \to (E, e₀) \) is a covering space so that \( p(π₁(\tilde{E}, x₀)) = G \) then there is an homeomorphism \( φ : (\tilde{E}, \tilde{x₀}) \to (\tilde{E}(G), \tilde{e₀}) \) so that \( p = p^G \circ φ \).

Now we apply this construction when we have two sections \( s₀, f₀ : (B, b₀) \to (E, e₀) \) of a fiber bundle \( q : (E, e₀) \to (B, b₀) \). For this we suppose that \( E, B \) and the fiber \( F₀ = q^{-1}(b₀) \) are compact spaces with \( B \) and \( E \) satisfying the hypothesis as in (1) above.

More precisely, we construct the universal covering spaces for the trivial subgroups \([\beta]₀ \subset π₁(B, b₀)\) and \([\tilde{π}₀] \subset π₁(E, e₀)\) which are denoted by \( p^{\tilde{B}} : \tilde{B}(b₀) \to B \) and \( p^{\tilde{E}} : \tilde{E}(e₀) \to E \). We also consider the regular covering space \( p^{\tilde{F₀}} : \tilde{E}(F₀) \to E \) where \( \tilde{E}(F₀) = \tilde{E}(π₁(F₀, e₀)) \).

As in (2) above we denote \( p^{\tilde{F₀}}_e : \tilde{E}(e₀) \to \tilde{E}(F₀) \) for the covering space so that \( p^G = p^{\tilde{F₀}} \circ p^{\tilde{F₀}}_e \).

From these constructions it is easy to explicit the covering projections, that is \( p^{\tilde{B}}(b, β) = b, p^{\tilde{E}}(e, β) = e \) and \( p^{\tilde{F₀}}_e(e, α) = (e, α)F₀ ∈ \tilde{E}(F₀) \). Moreover from
the sections \( s_0, f_0 : (B, b_0) \to (E, e_0) \) it is possible to explicit two special lifting maps as in the following lemma:

**Lemma 2.1.** The maps

\[
\tilde{s}_0, \tilde{f}_0 : (\tilde{B}(b_0), \tilde{b}_0) \to (\tilde{E}(e_0), \tilde{e}_0) \quad \text{and} \quad s_{F_0}, f_{F_0} : (\tilde{B}(b_0), \tilde{b}_0) \to (\tilde{E}(F_0), \tilde{e}_0)
\]

given by \( \tilde{s}_0(b, \beta)_{b_0} = \langle s_0(b), s_0(\beta) \rangle_{e_0}, \) \( \tilde{f}_0(b, \beta)_{b_0} = \langle f_0(b), f_0(\beta) \rangle_{e_0}, \) \( s_{F_0}(b, \beta)_{b_0} = \langle s_0(b), s_0(\beta) \rangle_{F_0} \) and \( f_{F_0}(b, \beta)_{b_0} = \langle f_0(b), f_0(\beta) \rangle_{F_0} \) are continuous and the following diagram commutes.

**Proof:** The commutativity is immediate from the constructions. Note that the continuity of the maps is given by the choice of the topology on the sets \( \tilde{B}(b_0), \tilde{E}(e_0) \) and \( \tilde{E}(F_0) \). In fact, let \( \langle b, \beta \rangle_{b_0} \in \tilde{B}(b_0) \) and \( V (\langle s_0(b), s_0(\beta) \rangle_{e_0}) \) be a basic open set of the topology on \( \tilde{E}(e_0) \) where \( V \) is an open neighborhood of \( s_0(b) \) in \( E \). From the continuity of \( s_0 \) let \( U = \tilde{s}_0^{-1}(V) \subseteq B \) an open neighborhood of \( b \) on \( B \) and note that \( \tilde{s}_0(U (\langle b, \beta \rangle_{b_0})) \subseteq V (\langle s_0(b), s_0(\beta) \rangle_{e_0}) \).

The continuity of the \( \tilde{f}_0, s_{F_0} \) and \( f_{F_0} \) is shown by similar argument. \( \Box \)

For \( \beta \in \pi_1(B, b_0) \) the correspondent deck transformation we denote by \( T_{\beta} \in Cov(\tilde{B}(b_0)/B) \) and similarly, \( T_{\alpha}, T_{\gamma} \) for \( \alpha \in \pi_1(E, e_0) \) and \( \gamma \in \pi_1(F_0, e_0) \lt \pi_1(E, e_0) \).

From the covering map constructions we can to explicit the following fibers, where we use the same symbol to express the loop path and its class on fundamental groups:

\[
(p_{F_0})^{-1}((e_0, \pi_0)_{F_0}) = (p_{F_0})^{-1}(\pi_{F_0})^{-1}(\pi_{F_0})^{-1}(\pi_{F_0})^{-1}(e_0) = \{ (e_0, s_0(\beta))_{F_0} = (e_0, f_0(\beta))_{F_0} ; \beta \in \pi_1(B, b_0) \} ;
\]

We know that on thoses fibers we have a right transitive action of the fundamental group and a left action of the deck transformation group. For example, if \( \beta \in \pi_1(B, b_0) \) then

\[
T_{\beta} (\langle b_0, \beta_1 \rangle_{b_0}) = \langle b_0, \beta_1 \rangle_{b_0} \ast \beta^{-1} = \langle b_0, \beta_1 \ast \beta^{-1} \rangle_{b_0} ,
\]

\( (2.1) \)
Lemma 2.3. \( \text{Cov}(E(e_0)/E(F_0)) \simeq \pi_1(F_0,e_0) \) and \( \text{Cov}(E(F_0)/E) \simeq \pi_1(B,b_0) \simeq \text{Cov}(\tilde{B}(b_0)/B) \).

\[ \text{Lemma 2.4.} \]

1. If \( \tilde{e} \in \mathcal{L}(s_0) \) and \( \tilde{e} = \tilde{s}_0 \circ T_\beta \) then there is only one \( \alpha(\tilde{e}) \in \pi_1(E_0,e_0) \) so that \( T_\alpha(\tilde{e}) = \tilde{s}_0 \circ T_\beta \) and moreover \( \alpha(\tilde{e}) = s_0(\beta) \).

2. If \( \tilde{f} \in \mathcal{L}(f_0) \) and \( \tilde{f} = \tilde{f}_0 \circ T_\beta \) then there is only one \( \alpha(\tilde{f}) \in \pi_1(E,e_0) \) so that \( T_\alpha(\tilde{f}) = \tilde{f}_0 \circ T_\beta \) and moreover \( \alpha(\tilde{f}) = f_0(\beta) \).

\[ \text{Remark 2.5.} \]

From lemmas 2.4 and 2.3 part (3), for each pair \( (\tilde{s},\tilde{f}) \in \mathcal{L}(s_0,f_0) \) we can write in the form

\[ (\tilde{s},\tilde{f}) = (T_{\alpha_1} \circ \tilde{s}_0,T_{\alpha_2} \circ \tilde{f}_0) = (T_{\alpha_1 \cdot s_0(q(\alpha_1^{-1})) \cdot f_0(q(\alpha_2^{-1}))} \circ \tilde{s}_0,T_{\alpha_2 \cdot f_0(q(\alpha_2^{-1}))}) \circ \tilde{f}_0) = (T_{\alpha_1 \cdot s_0(q(\alpha_1^{-1}))} \circ \tilde{s}_0 \circ T_{q(\alpha_1)}, T_{\alpha_2 \cdot f_0(q(\alpha_2^{-1}))} \circ \tilde{f}_0 \circ T_{q(\alpha_2)}) \]
where $\gamma_1 = \alpha_1 \ast s_0(q(\alpha_1^{-1})) \in \pi_1(F_0, e_0)$ and $\gamma_2 = \alpha_2 \ast f_0(q(\alpha_2^{-1})) \in \pi_1(F_0, e_0)$. Because we are considering the constant homotopy on the basic space $T_B : B \times [0,1] \to B$ to deform the initial sections $s_0, f_0$ over $B$, so we assume that $q(\alpha_1) = q(\alpha_2)$. From this we consider only the pairs of liftings $(T_{\gamma_1}^{-1} \ast s_0, \tilde{f}_0) \in \mathcal{L}(s_0; f_{F_0})$ or $(s_0, T_{\gamma_1}^{-1} \ast s_0 \circ f_0) \in \mathcal{L}(f_0; s_{F_0})$.

**Definition 2.6.**

1. Given $(\tilde{s}_1, \tilde{f}_0)$ and $(\tilde{s}_2, \tilde{f}_0) \in \mathcal{L}(s_0; f_{F_0})$ we say that $(\tilde{s}_1, \tilde{f}_0)$ is lifting related with $(\tilde{s}_2, \tilde{f}_0)$ for the $f_0$, in symbols $\tilde{s}_1 R_{\tilde{s}_0} \tilde{s}_2$, or $(\tilde{s}_1, \tilde{f}_0) R_{\tilde{s}_0}(\tilde{s}_2, \tilde{f}_0)$, if and only if $T_{\tilde{s}_0} R_{\tilde{s}_0} \tilde{s}_1 = \tilde{s}_2 \circ T_{\tilde{s}_0}$ for some $\beta \in \pi_1(B, b_0)$.

2. Similarly for the elements $(\tilde{s}_0, \tilde{f}_1) (\tilde{s}_0, \tilde{f}_2) \in \mathcal{L}(f_0; s_{F_0})$ we define the relation $R_{s_0}$ by $f_1 R_{s_0} f_2 \Leftrightarrow T_{s_0} \circ \tilde{f}_1 = \tilde{f}_2 \circ T_{\tilde{s}_0}$ for some $\beta \in \pi_1(B, b_0)$.

**Proposition 2.7.**

1. The relation $R_{f_0}$ is an equivalence relation on the set $\mathcal{L}(s_0; f_{F_0})$.

2. The relation $R_{s_0}$ is an equivalence relation on the set $\mathcal{L}(f_0; s_{F_0})$.

**Proof:** If $\beta = [b_0] \in \pi_1(B, b_0)$ then $\tilde{s}_1 R_{\tilde{s}_0} \tilde{s}_1$. If $\tilde{s}_1 R_{f_0} \tilde{s}_2$ with $T_{f_0} R_{f_0} \tilde{s}_1 = \tilde{s}_2 \circ T_{f_0}$ thus $T_{f_0} R_{f_0} \tilde{s}_1 = \tilde{s}_2 \circ T_{f_0}$. If $\tilde{s}_1 R_{f_0} \tilde{s}_2$ and $\tilde{s}_2 R_{f_0} \tilde{s}_3$ which implies that there are $\beta_1, \beta_2 \in \pi_1(B, b_0)$ such that $T_{f_0} R_{f_0} \tilde{s}_1 = \tilde{s}_2 \circ T_{f_0}$ and $T_{f_0} R_{f_0} \tilde{s}_2 = \tilde{s}_3 \circ T_{f_0}$. Therefore,

$$T_{f_0} R_{f_0} \tilde{s}_1 = T_{f_0} R_{f_0} \tilde{s}_2 \circ T_{f_0} = T_{f_0} R_{f_0} \tilde{s}_3 \circ T_{f_0} = T_{f_0} R_{f_0} \tilde{s}_4 \circ T_{f_0} = \vdots$$

The proof of (2) is analogous. $\square$

Let $R_{f_0}(\mathcal{L}(s_0; f_{F_0}))$ and $R_{s_0}(\mathcal{L}(f_0; s_{F_0}))$ be the quotient spaces by the relations $R_{f_0}$ and $R_{s_0}$ on the spaces $\mathcal{L}(s_0; f_{F_0})$ and $\mathcal{L}(f_0; s_{F_0})$ respectively. Denote by $r_{f_0}(\mathcal{L}(s_0; f_{F_0}))$ and $r_{s_0}(\mathcal{L}(f_0; s_{F_0}))$ the respective cardinals of the quotient spaces.

The following definition is approximation between the relation $R_{f_0}$, or $R_{s_0}$, and the Reidemeister relation relative to the subgroup $\pi_1(F_0, e_0)$ as we will view in the next section.

**Definition 2.8.** Let $\tilde{s}_1 = T_{\gamma_1} \circ \tilde{s}_0, \tilde{s}_2 = T_{\gamma_2} \circ \tilde{s}_0$ be in $\mathcal{L}(s_0; f_{F_0})$ where $\gamma_1, \gamma_2 \in \pi_1(F_0, e_0)$. We say that $\tilde{s}_1$ is lifting related with $\tilde{s}_2$, in symbol $\tilde{s}_1 R_{\tilde{s}_0} \tilde{s}_2$, if there is $\beta \in \pi_1(B, b_0)$ so that $f_0(\beta) \circ \gamma_1 = \gamma_2 \circ f_0(\beta)$. Similarly we define for $\tilde{f}_1 = T_{\gamma_1} \circ \tilde{f}_0, \tilde{f}_2 = T_{\gamma_2} \circ \tilde{f}_0$ be in $\mathcal{L}(f_0; s_{F_0})$. That is $\tilde{f}_1 R_{\tilde{s}_0} \tilde{f}_2$ if and only if there is $\beta \in \pi_1(B, b_0)$ such that $f_0(\beta) \circ \gamma_1 = \gamma_2 \circ s_0(\beta)$.

**Theorem 2.9.**
1. The relation $R_L$ defined on $\mathcal{L}(s_0; f_{E_0})$ is an equivalence relation.

2. The relation $R_L$ defined on $\mathcal{L}(f_0; s_{E_0})$ is an equivalence relation.

3. $[\tilde{s}]_{f_0} = [\tilde{s}]_{L}$ and $[\tilde{f}]_{s_0} = [\tilde{f}]_{L}$. Therefore if $R_L(\mathcal{L}(s_0; f_{E_0}))$ and $R_L(\mathcal{L}(f_0; s_{E_0}))$ are the quotient set by the relation $R_L$ then there is an one to one correspondence between the followings sets:

$$R_{f_0}(\mathcal{L}(s_0; f_{E_0})) \leftrightarrow R_L(\mathcal{L}(s_0; f_{E_0})) \leftrightarrow R_{s_0}(\mathcal{L}(f_0; s_{E_0})) \leftrightarrow R_L(\mathcal{L}(f_0; s_{E_0})).$$

**Proof:** We will prove the item (1). Obviously the relation $R_L$ is reflexive and symmetric. If $\tilde{s}_i = T_{r_1} \circ \tilde{s}_{r_2}$ for $i = 1, 2, 3$ and $\tilde{s}_1R_L\tilde{s}_2$ and $\tilde{s}_3R_L\tilde{s}_3$ then there exists $\beta_1$ and $\beta_2$ in $\pi_1(B, b_0)$ such that $f_0(\beta_1) * \gamma_1 = \gamma_2 * s_0(\beta_1)$ and $f_0(\beta_2) * \gamma_2 = \gamma_3 * s_0(\beta_2)$. So we have

$$f_0(\beta_2 * \beta_1) * \gamma_1 = f_0(\beta_2) * (f_0(\beta_1) * \gamma_1) = f_0(\beta_2) * \gamma_2 * s_0(\beta_1) = \gamma_3 * s_0(\beta_2) * s_0(\beta_1).$$

Therefore $\tilde{s}_1R_L\tilde{s}_3$. The proof of (2) is analogous.

In fact $[\tilde{s}]_{f_0} = [\tilde{s}]_{L}$. If $\tilde{s}_1R_{f_0}\tilde{s}_2$, then there is $\beta \in \pi_1(B, b_0)$ such that $T_{f_0(\beta)} \circ \tilde{s}_1 = \tilde{s}_2 \circ T_{\beta}$. But $\tilde{s}_1, \tilde{s}_2 \in \mathcal{L}(s_0; f_{E_0})$ so there are $\gamma_1, \gamma_2 \in \pi_1(F_0, c_0)$ such that $\tilde{s}_1 = T_{\gamma_1} \circ \tilde{s}_0$ and $\tilde{s}_2 = T_{\gamma_2} \circ \tilde{s}_0$. Since $\tilde{s}_1R_{f_0}\tilde{s}_2$ we have:

$$T_{f_0(\beta)} \circ \tilde{s}_1 = \tilde{s}_2 \circ T_{\beta},$$

$$T_{f_0(\beta)} \circ T_{\gamma_1} \circ \tilde{s}_0 = T_{\gamma_2 \circ s_0(\beta)} \circ \tilde{s}_0.$$

The last equation means that $\tilde{s}_1R_L\tilde{s}_2$. Therefore there is an one to one correspondence between the sets. The second part is analogous. \hfill \Box

3. **Algebraic Reidemeister classes relative of a subgroup**

**Definition 3.1.** Let $\psi, \varphi : G_1 \to G_0$ be group homomorphisms and $H_0$ a subgroup of $G_0$. We say that two elements $h_1, h_2 \in H_0$ are $(\varphi, \psi; H)$—algebraic Reidemeister related, in symbols $h_1R_{(\varphi, \psi; H_0)}h_2 = h_1R_{H_0}h_2$ or $h_1R_AP_{H_0}$ if there is $g \in G_0$ such that $\varphi(g)h_1 = h_2\psi(g)$.

It is easy to prove that $R_{(\varphi, \psi; H_0)}$ is an equivalence relation on $H_0$, called the algebraic Reidemeister relation of $\varphi$ and $\psi$ relative to the subgroup $H_0$. We denoted by $[h]_{(\varphi, \psi; H_0)} = [h]_{H_0}$ or $[h]_A$ the algebraic Reidemeister class determined by $h \in H_0$ and by $A(\varphi, \psi; H_0)$ to the quotient set. The cardinal of $A(\varphi, \psi; H_0)$ which is indicated by $r(\varphi, \psi; H_0)$ is called $(\varphi, \psi; H_0)$—Reidemeister number. When $H_0 = G_0$ we denoted $R(\varphi, \psi; G_0) = R(\varphi, \psi)$ and $r(\varphi, \psi; G_0) = r(\varphi, \psi)$.

**Proposition 3.2.** Let $\varphi, \psi : G_1 \to G_0$ be homomorphisms and $H_0, K_0$ subgroups of $G_0$. If $H_0 \leq K_0$ then $r(\varphi, \psi; H_0) \leq r(\varphi, \psi; K_0)$. 

Proof: Just set the injection $R_A(\varphi, \psi; H_0) \leftrightarrow R_A(\varphi, \psi; K_0)$, $[a]_{H_0} \mapsto [a]_{K_0}$. □

Remark 3.3. If \{e_{G_0}\} is the trivial subgroup of $G_0$ then for any subgroup $H_0$ of $G_0$ we have $1 = r(\varphi, \psi; \{e_{G_0}\}) \leq r(\varphi, \psi; H_0) \leq r(\varphi, \psi)$.

Proposition 3.4. Let $\varphi, \psi : G_2 \to G_1$ be homomorphisms, $K_1 \leq G_1$ and $\Phi : G_1 \to G_0$ a homomorphism with $H_0 = \Phi(K_1)$. The following map $\Phi_A : R_A(\varphi, \psi; K_1) \to R_A(\Phi \circ \varphi, \Phi \circ \psi; H_0)$ given by $\Phi_A([k]_{K_1}) = [\Phi(k)]_{H_0}$ is surjective. Therefore $r(\varphi, \psi; K_1) \geq r(\Phi \circ \varphi, \Phi \circ \psi; H_0)$.

If $\Phi$ has the left inverse homomorphism $\Psi : G_0 \to G_1$ then $\Phi_A$ is an one to one correspondence and $\Phi_A^{-1}[\Phi(k)]_{H_0} = [k]_{K_1}$ so $r(\varphi, \psi; K_1) = r(\Phi \circ \varphi, \Phi \circ \psi; H_0)$.

Proof: If $[k_1]_{K_1} = [k_2]_{K_1}$ there is $g_2$ such that $\varphi(g_2)k_1 = k_2\varphi(g_2)$, then $\Phi(\varphi(g_2))\Phi(k_1) = \Phi(k_2)\Phi(\varphi(g_2))$. Therefore we have a well defined map $\Phi_A^1 : R_A(\varphi, \psi; K_1) \to R_A(\Phi \circ \varphi, \Phi \circ \psi; H_0)$ given by $[k]_{K_1} \mapsto [\Phi(k)]_{H_0}$. As $H_0 = \Phi(K_1)$, it is easy to prove that the map $\Phi_A$ is surjective.

Otherwise if $\Psi : G_0 \to G_1$ is a left inverse of $\Phi$ then when we apply $\Psi$ in the equation $\Phi(\varphi(g_2))\Phi(k_1) = \Phi(k_2)\Phi(\varphi(g_2))$ we have a well defined map $\Psi_{\Phi(K_0)} : R_A(\Phi \circ \varphi, \Phi \circ \psi; K_1) \to R_A(\varphi, \psi; K_1)$ such that $\Phi_A \circ \Psi_{\Phi(K_0)}$ is identity of $R_A(\varphi, \psi; K_1)$.

Therefore $\Phi_A$ is an one to one correspondence and we have the equivalence on the Reidemeister numbers $r(\varphi, \psi; K_1) = r(\Phi \circ \varphi, \Phi \circ \psi; H_0)$ with $H_0 = \Phi(K_1)$. □

Example 3.5 (Case trivial fiber bundle). Let $f, g : (B, b_0) \to (F, y_0)$ be continuous maps. So we have $f, g : \pi_1(B, b_0) \to \pi_1(F, y_0)$ and the set of algebraic Reidemeister classes $R_A(f, g)$. Now we consider the trivial fiber bundle $q : (B \times F, (b_0), y_0)) \to (B, b_0)$ so the maps $f, g$ induce two sections $s_f, s_g : (B, b_0) \to (B \times F, (b_0), y_0)$ given by $s_f(b) = (b, f(b))$ and $s_g(b) = (b, g(b))$. Let $F_0 = \{b_0\} \times F = q^{-1}(b_0)$ be the fiber over $b_0$ with base point $e_0 = (b_0, y_0)$ so $\pi_1(F_0)$. Then we can consider the algebraic classes of Reidemeister $R_A(s_g, s_f; \pi_1(F_0, e_0))$ and $\Phi(\pi_1(F_0, b_0), y_0)) \equiv \pi_1(F, y_0)$. Then we conclude that:

\[
R_A(s_g, s_f; \pi_1(F_0, e_0)) \leftrightarrow R_A(\Phi \circ s_g, \Phi \circ s_f; \Phi(\pi_1(F_0, e_0))) \leftrightarrow R_A(g, f; \pi_1(F, y_0)) = R_A(g, f) \tag{3.1}
\]

Example 3.6 (Case not trivial fiber bundle). We consider two sections $s_0, f_0 : (B, b_0) \to (E, e_0)$ of the fiber bundle $q : (E, e_0) \to (B, b_0)$. We used $s_0$ to describe the structure of the group $\pi_1(E, e_0)$ as the semi direct product $\pi_1(F_0, e_0) \rtimes \pi_1(B, b_0)$. Formally, let $\Phi : \pi_1(E, e_0) \to \pi_1(F_0, e_0) \rtimes \pi_1(B, b_0)$ be the isomorphism given by

\[
\Phi(\alpha) = (\alpha \ast s_0(q(\alpha^{-1})), q(\alpha)) \in \pi_1(F_0, e_0) \rtimes \pi_1(B, b_0), \tag{3.2}
\]

The operation on $\pi_1(F_0, e_0) \rtimes \pi_1(B, b_0)$ is expressed by

\[
(\gamma_1, \beta_1) \bullet (\gamma_2, \beta_2) := (\gamma_1 \ast s_0(\beta_1) \ast \gamma_2 \ast s_0(\beta_1^{-1})), \beta_1 \ast \beta_2). \tag{3.3}
\]
Let $\Psi := \Phi^{-1} : \pi_1(F_0, e_0) \times \pi_1(B, b_0) \to \pi_1(E, e_0)$ be the inverse isomorphism of $\Phi$ given by $\Psi(\gamma, \beta) = \gamma \ast s_0(\beta^{-1})$ and let $H_0 = \pi_1(F_0, e_0) \times \{[b_0]\} = \Phi(\pi_1(F_0, e_0))$ be the subgroup of $\pi_1(F_0, e_0) \times \pi_1(B, b_0)$. By the proposition 3.4 we have $R_A(s_0, f_0; \pi_1(F_0, e_0)) \leftrightarrow R_A(\Phi \circ s_0, \Phi \circ f_0; H_0)$.

For the operation $\bullet$ in $\pi_1(F_0, e_0) \times \pi_1(E, e_0)$ expressed in (3.3) when we describe the classes of $R_A(\Phi \circ s_0, \Phi \circ f_0; H_0)$ we have the same classes on

$$R_A(s_0, f_0; \pi_1(F_0, e_0))$$

$$\Phi \circ s_0(\beta) \bullet (\gamma_1, [b_0]) = (\gamma_2, [b_0]) \bullet (\Phi \circ f_0(\beta))$$

$$(s_0(\beta) \ast \gamma_1 \ast s_0(\beta^{-1}), \beta) = (\gamma_2 \ast f_0(\beta) \ast s_0(\beta^{-1}), \beta)$$

$$(3.4)$$

4. The coincidence set and the Nielsen classes for sections on the fiber bundle

Let $s_0, f_0 : (B, b_0) \to (E, e_0)$ be the sections of a fiber bundle $q : (E, e_0) \to (B, b_0)$ and $\Gamma^B_E(s_0, f_0) = \{b \in B, s_0(b) = f_0(b) \neq \emptyset\}$ be the coincidence topological space induced from $B$. Note that $\Gamma^B_E(s_0, f_0) = s_0^{-1}(f_0(B)) = f_0^{-1}(s_0(B))$.

In $\Gamma^B_E(s_0, f_0)$ we defined the Nielsen classes for $b_1, b_2 \in \Gamma^B_E(s_0, f_0)$ saying that $b_1$ is Nielsen related to $b_2$, in symbols $b_1 \sim b_2$, if and only if there is a path $\beta_{b_1}^{b_2}$ on $B$ connecting $b_1$ to $b_2$ such that $s_0(\beta_{b_1}^{b_2})$ is homotopic to $f_0(\beta_{b_1}^{b_2})$ relative to $\{0, 1\}$. It easy is to verify that $N$ is an equivalent relation and we denote by $[b_1]_N$ the class determined by $b_1$. If $\tilde{\Gamma}^B_E(s_0, f_0)$ is the quotient set of $\Gamma^B_E(s_0, f_0)$ by the Nielsen relation, we denote by $p_N : \Gamma^B_E(s_0, f_0) \to \tilde{\Gamma}^B_E(s_0, f_0)$ the canonical projection map.

Considering $\tilde{\Gamma}^B_E(s_0, f_0)$ with the topology co-induced by $p_N$ we have the following statements.

**Theorem 4.1.**

1. If $[b_1]_{cc}$ is the connected component by path of $b_1 \in \Gamma^B_E(s_0, f_0)$ then $[b_1]_{cc} \subset [b_1]_N$.

2. If $E$ is a Hausdorff topological space then $\Gamma^B_E(s_0, f_0)$ is closed in $B$.

3. If $B$ is locally path connected and $E$ is Hausdorff and semilocally 1-connected topological space then $\tilde{\Gamma}^B_E(s_0, f_0)$ is discrete topological space.

4. If $B$ and $E$ satisfies the before conditions and $\Gamma^B_E(s_0, f_0)$ is compact then $\tilde{\Gamma}^B_E(s_0, f_0)$ is finite.

**Proof:** The (1), (2) and (4) items are easy to prove. We will prove only the item (3). Let $b_2 \in [b_1]_N$ and consider an open set $V_{b_2}$ such that $i : \pi_1(V_{b_2}, c_2) \to \pi_1(E, c_2)$ is trivial homomorphism. Now $W_{b_2} = s_0^{-1}(V_{b_2}) \cap f_0(V_{b_2}) \cap U_{b_2}$ where $U_{b_2}$ is connected path neighborhood of $b_2$. It is immediate to verify that $W_{b_2} \subset [b_1]_N$ so
Theorem 4.3. For the first part we consider the following diagram:

\[ \Gamma_E^B(s_0, f_0) \]

Proof: For the first part we consider \( \beta = \beta_{b_1}(2) \ast (\beta_{b_1}(1))^{-1} \). Now \( P_R(b_1, \beta_{b_1}(1)) = [[s_0(\beta_{b_1}(1) \ast f_0(\beta_{b_1}(1)))^{-1}]_A \]
\[ = [[s_0(\beta) \ast (s_0(\beta_{b_1}(1) \ast f_0(\beta_{b_1}(1)))^{-1}) \ast f_0(\beta)^{-1}]_A \]
\[ = P_R(b_1, \beta_{b_1}(2)) \]

For the second part, if \( (b_1, \beta_{b_1}(1)), (b_2, \beta_{b_2}(2)) \in \mathcal{B} \) and \( [b_1]_N = [b_2]_N \) on \( \Gamma_E^B(s_0, f_0) \) then there is a path \( \beta_{b_2}(N) \) between \( b_1 \) and \( b_2 \) such that \( f_0(\beta_{b_2}(N)) \) is homotopic to \( s_0(\beta_{b_2}(N)) \) relative to \( [0, 1] \). So we have:
\[ P_R(b_1, \beta_{b_1}) = [[s_0(\beta_{b_1}) \ast f(\beta_{b_1})^{-1}]_A \]
\[ = [[s_0(\beta_{b_1}) \ast s_0(\beta_{b_2}(N)) \ast f(\beta_{b_2}(N))^{-1}]_A \]
\[ = [[s_0(\beta_{b_1}) \ast f_0(\beta_{b_2})^{-1}]_A = P_R(b_2, \beta_{b_2}) \]

So \( P_R([b_1]) = P_R(b_1, \beta_{b_1}) \) is a well-defined map as on the commutative diagram and it is easy to see that \( P_R \) is an injection.

\[ \square \]

Theorem 4.4. Let \( \Gamma_{E(co)}^{\tilde{E}(co)}(\tilde{s}_0, \tilde{f}_1) \) and \( \Gamma_{E(co)}^{\tilde{E}(co)}(\tilde{s}_0, \tilde{f}_2) \) be the coincidence set for \( \tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(f_0; s_{f_0}) \).

1. If \( [\tilde{f}_1]_s = [\tilde{f}_2]_s \) then \( p^{\tilde{E}(co)}(\Gamma_{E(co)}^{\tilde{E}(co)}(\tilde{s}_0, \tilde{f}_1)) = p^{\tilde{E}(co)}(\Gamma_{E(co)}^{\tilde{E}(co)}(\tilde{s}_0, \tilde{f}_2)) \).
2. If $p^{b_0}\left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1)\right) \cap p^{b_0}\left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2)\right) \neq \emptyset$ then $[\tilde{f}_1]_{s_0} = [\tilde{f}_2]_{s_0}$.

**Proof:** (1). Since $[\tilde{f}_1]_{s_0} = [\tilde{f}_2]_{s_0}$ there is $\tilde{b} \in \pi_1(B, b_0)$ which satisfies $T_{\pi_1(\tilde{b})} \circ \tilde{f}_1 = \tilde{f}_2 \circ T_{\tilde{b}}$. If $\tilde{b} \in \left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1)\right)$ then $\tilde{s}_0(\tilde{b}) = \tilde{f}_1(\tilde{b})$, so we have

$$\tilde{s}_0 \circ T_{\tilde{b}}(\tilde{b}) = T_{\pi_1(\tilde{b})} \circ \tilde{s}_0(\tilde{b}) = T_{\pi_1(\tilde{b})} \circ \tilde{f}_1(\tilde{b}) = \tilde{f}_2 \circ T_{\tilde{b}}(\tilde{b}).$$

Therefore $T_{\tilde{b}}(\tilde{b}) \in \Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2)$. The verification of the inverse inclusion is analogous. Since $T_{\tilde{b}}$ established an one to one correspondence between $\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1)$ and $\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2)$ then when we apply $p^{b_0}$ we have

$$p^{b_0}\left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1)\right) = p^{b_0}\left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2)\right).$$

Since (3) is equivalent to (2) it is sufficient to prove the item (2). If

$$p^{b_0}\left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1)\right) \cap p^{b_0}\left(\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2)\right) \neq \emptyset,$$

then $\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1) \neq \emptyset$ and $\Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2) \neq \emptyset$. Then there are $\tilde{b}_1 \in \Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_1)$ and $\tilde{b}_2 \in \Gamma^{\tilde{B}(b_0)}_{E(c_0)}(\tilde{s}_0, \tilde{f}_2)$ such that $p^{b_0}(\tilde{b}_1) = p^{b_0}(\tilde{b}_2) := b$. Since the action of the fundamental group $\pi_1(B, b_0)$ on the fibers is transitive, there is $\tilde{b} \in \pi_1(B, b_0)$ such that $T_{\tilde{b}}(\tilde{b}_1) = \tilde{b}_2$ and

$$\tilde{f}_2 \circ T_{\tilde{b}}(\tilde{b}_1) = \tilde{s}_0(\tilde{b}_1) = T_{\pi_1(\tilde{b})} \circ \tilde{s}_0(\tilde{b}_1) = T_{\pi_1(\tilde{b})} \circ \tilde{f}_1(\tilde{b}_1)$$

Since $\tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(f_0; s_{F_0})$ and the coincidence occurs in $\tilde{b}_1$ it follows that $\tilde{f}_2 \circ T_{\tilde{b}} = T_{\pi_1(\tilde{b})} \circ \tilde{f}_1$ as the bellow diagram. Therefore we have $[\tilde{f}_2]_{s_0} = [\tilde{f}_1]_{s_0}$.

\[ \begin{array}{ccc}
(\tilde{B}(b_0), \tilde{b}_2) & \xrightarrow{T_{\tilde{b}}} & (\tilde{E}(c_0), \tilde{e}) \\
\downarrow & & \downarrow \xrightarrow{T_{\pi_1(\tilde{b})}} \\
(\tilde{B}(b_0), \tilde{b}_1) & \xrightarrow{s_{F_0}} & (\tilde{E}(F_0), s_{F_0}(\tilde{b}_1)) \\
\end{array} \]

\[ \begin{array}{ccc}
(\tilde{E}(c_0), \tilde{e}) & \xrightarrow{T_{\pi_1(\tilde{b})}} & (\tilde{E}(c_0), \tilde{f}_1(\tilde{b}_1)) \\
\end{array} \]
Theorem 4.4. Let $\Gamma_{E(e_0)}^B(b_0)(\tilde{f}_0, \tilde{s}_1)$ and $\Gamma_{E(e_0)}^B(b_0)(\tilde{f}_0, \tilde{s}_2)$ be the coincidence set for $\tilde{s}_1, \tilde{s}_2 \in \mathcal{L}(s_0; f_{F_0})$.

1. If $[\tilde{s}_1]_{f_0} = [\tilde{s}_2]_{f_0}$ then $p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{f}_0, \tilde{s}_1)) = p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{f}_0, \tilde{s}_2))$.

2. If $p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{f}_0, \tilde{s}_1)) \cap p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{f}_0, \tilde{s}_2)) \neq \emptyset$ then $[\tilde{s}_1]_{f_0} = [\tilde{s}_2]_{f_0}$.

Now, $[\tilde{s}_1]_{f_0} = [\tilde{s}_2]_L \in R_L(\mathcal{L}(s_0; f_{F_0})$ by theorem 2.9. If $\tilde{s}_1 = T_{\gamma_1} \circ \tilde{s}_0$ and $\tilde{s}_2 = T_{\gamma_2} \circ \tilde{s}_0$ with $\gamma_1, \gamma_2 \in \pi_1(F_0, e_0)$ then, by definition 2.8, we have $[T_{\gamma_1} \circ \tilde{s}_0]_L = [T_{\gamma_2} \circ \tilde{s}_0]_L$ if and only if $[\gamma_1]_A = [\gamma_2]_A \in R_A(s_0, f_0; \pi_1(F_0, e_0))$. From this, it follows the main theorem:

Theorem 4.5. Let $\Gamma_{E(e_0)}^B(b_0)(\tilde{s}_0, \tilde{f}_1)$ and $\Gamma_{E(e_0)}^B(b_0)(\tilde{s}_0, \tilde{f}_2)$ be the coincidence set for $\tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(f_0; s_{F_0})$.

1. There is an one to one correspondence

   $\Psi : R_L(\mathcal{L}(f_0; s_{F_0})) \rightarrow R_A(f_0, s_0; \pi_1(F_0, e_0)).$

2. If $[\tilde{f}_1]_L = [\tilde{f}_2]_L$ then $p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{s}_0, \tilde{f}_1)) = p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{s}_0, \tilde{f}_2))$.

3. If $p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{s}_0, \tilde{f}_1)) \cap p_{b_0}^b(\Gamma_{E(e_0)}^B(b_0)(\tilde{s}_0, \tilde{f}_2)) \neq \emptyset$ then $[\tilde{f}_1]_L = [\tilde{f}_2]_L$.

Remark 4.6. Note that the theorem follows from the theorems 2.1 and 4.3, and is true if we replace $f_0, f_1, f_2$ by $s_0, \tilde{s}_1, \tilde{s}_2$ and $s_{F_0}$ by $f_{F_0}$.

References

Reidemeister Classes for Coincidences Between Sections of a Fiber Bundle

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