Spectral Mapping Theorem for $C_0$-Semigroups of Drazin Spectrum

Abdelaziz Tajmouati and Hamid Boua

ABSTRACT: Let $(T(t))_{t \geq 0}$ be a $C_0$ semigroup of bounded linear operators on a Banach space $X$ and denote its generator by $A$. A fundamental problem to decide whether the Drazin spectrum of each operator $T(t)$ can be obtained from the Drazin spectrum of $A$. In particular, one hopes that the Drazin Spectral Mapping Theorem holds, i.e., $e^{\sigma_D(A)} = \sigma_D(T(t))\{0\}$ for all $t \geq 0$.

Key Words: Drazin invertibility, Spectrum Drazin, Semigroup of operators.

Contents

1 Introduction 63

2 Main results 64

1. Introduction

Throughout this work, $X$ denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on $X$. Let $A$ be a closed operator with domain $D(A)$, we denote by $R(A)$, $N(A)$, $p(A)$, $\sigma(A)$, $\sigma_r(T)$ and $\sigma_p(A)$ respectively the range, the kernel, the resolvent set, the spectrum, the residual spectrum and the point spectrum of $A$. The ascent of $A$ is defined by $a(A) = \min\{p : N(A^p) = N(A^{p+1})\}$, if no such $p$ exists, we let $a(A) = \infty$. Similarly, the descent of $A$ is $d(A) = \min\{q : R(A^q) = R(A^{q+1})\}$, if no such $q$ exists, we let $d(A) = \infty$ (see [7] and [8]). It is well known that if $A$ is bounded, and if both $a(A)$ and $d(A)$ are finite then $a(A) = d(A)$ and therefore we have the decomposition $X = R(A^p) \oplus N(A^p)$ where $p = a(A) = d(A)$. The descend and ascent spectrum are defined by $\sigma_{desc}(A) = \{\lambda \in \mathbb{C} : d(\lambda - A) = \infty\}$ and $\sigma_{asc}(A) = \{\lambda \in \mathbb{C} : a(\lambda - A) = \infty\}$. Recall that $A$ is a Drazin invertible if $p = a(A) = d(A) < \infty$ and $R(A^p)$ is closed. The Drazin spectrum is defined by $\sigma_D(A) = \{\lambda \in \mathbb{C} : \lambda - A$ not Drazin invertible $\}.$

A strongly continuous semigroup $(T(t))_{t \geq 0}$ is called eventually norm continuous, if there exists $t_0 \geq 0$ such that the function $t \mapsto T(t)$ is norm continuous from $(t_0, \infty)$ into $\mathcal{B}(X)$. Let $\Delta = \{z \in \mathbb{C} : \alpha_1 < \arg z < \alpha_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $(T(z))_{z \in \Delta}$ is an analytic semigroup in $\Delta$ if

(i) $z \mapsto T(z)$ is analytic in $\Delta$.

(ii) $T(0) = I$ and $\lim_{t \to 0} T(t)x = x$ for every $x \in X$. 

2010 Mathematics Subject Classification: 47A16, 47D06, 47D03.


Typeset by \texttt{\textsc{b}4\textsc{sp}} style.

© Soc. Paran. de Mat.
A semigroup \((T(t))_{t \geq 0}\) will be called analytic if it is analytic in some sector \(\Delta\) containing the nonnegative real axis.

A strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\) is called eventually differentiable if there exists \(t_0 \geq 0\) such that the orbit maps \(\xi_x : t \mapsto T(t)x\) are differentiable on \((t_0, \infty)\) for every \(x \in X\). The semigroup is called eventually compact, if there exists \(t_0 > 0\) such that \(T(t_0)\) is compact.

Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup on \(X\) with infinitesimal generator \(A\). We introduce the following operator acting on \(X\) and depending on the parameters \(\lambda \in \mathbb{C}\) and \(t \geq 0\), \(B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)xds, \ x \in X\). It is well known (see [5] and [10]) that \(B_\lambda(t)\) is a bounded linear operator on \(X\). Furthermore, for all \(n \in \mathbb{N}\), we have \((e^{\lambda t} - T(t))^n x = (\lambda - A)^n B_\lambda^n(t)x\), for all \(x \in X\) and \((e^{\lambda t} - T(t))^n x = B_\lambda^n(t)(\lambda - A)^n x\), for all \(x \in D(A^n)\). (See [4]).

In [9], Rainer Nagel and Jan Poland showed that, for an eventually norm continuous semigroup \((T(t))_{t \geq 0}\) with generator \(A\) one has \(e^{t\sigma(A)} = \sigma(T(t))\setminus\{0\}\) for all \(t \geq 0\). Rainer Nagel in [5] proved for the generator \(A\) of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\), we have the identities: \(e^{t\sigma(A)} = \sigma_p(T(t))\setminus\{0\}\) and \(e^{t\sigma_e(A)} = \sigma_e(T(t))\setminus\{0\}\) for all \(t \geq 0\). These works push to ask the following question: Does this spectral inclusion hold for the other parts of spectrum?

In this work, we show that this spectral mapping theorem of \(C_0\)-semigroups eventually norm continuous holds for Drazin spectrum.

2. Main results

We start by the following lemmas.

**Lemma 2.1.** Let \((T(t))_{t \geq 0}\) a \(C_0\)-semigroup on \(X\) with infinitesimal generator \(A\). For all \(\lambda \in \mathbb{C}\) and \(t \geq 0\), there exists \(F_\lambda(t), G_\lambda(t) \in \mathcal{B}(X)\) such that

1. \(\forall x \in X\), \(F_\lambda(t)x \in D(A)\) and \((\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI,\)

2. The operators \(\lambda - A, F_\lambda(t), G_\lambda(t)\) and \(B_\lambda(t)\) are are mutually commuting.

**Proof:**

1. For every \(\lambda \in \mathbb{C}\) and \(t \geq 0\), let \(F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(s)xds\). \(F_\lambda(t)\) is a bounded
linear operator on $X$. Moreover for every $x \in X$, we have

$$\frac{T(h) - I}{h} F_\lambda(t)x = \frac{T(h) - I}{h} \int_0^t e^{-\lambda s} B_\lambda(s) x ds$$

$$= \frac{1}{h} \int_0^t \int_0^s e^{-\lambda u} T(u + h) x du ds$$

$$- \frac{1}{h} \int_0^t \int_0^s e^{-\lambda u} T(u) x du ds$$

$$= \frac{1}{h} \int_0^t \left( \int_0^s e^{-\lambda u} T(u + h) x du ight) ds$$

$$- \int_0^t e^{-\lambda u} T(u) x du$$

$$= \int_0^t \left( \frac{\lambda h}{h} \int_h^s e^{-\lambda u} T(u) x du ight) ds$$

$$- \frac{1}{h} \int_0^s e^{-\lambda u} T(u) x du ds$$

$$= \int_0^t \left( \frac{\lambda h}{h} \int_h^s e^{-\lambda u} T(u) x du ight) ds$$

$$+ \frac{\lambda h}{h} \int_0^s e^{-\lambda u} T(u) x du$$

$$- \frac{1}{h} \int_0^h e^{-\lambda u} T(u) x du ds.$$
Therefore

\[ F_\lambda(t)B_\lambda(t)x = \int_0^t e^{-\lambda u}B_\lambda(u)B_\lambda(t)x \, du \]
\[ = \int_0^t e^{-\lambda u}B_\lambda(t)B_\lambda(u)x \, du \]
\[ = B_\lambda(t)\int_0^t e^{-\lambda u}B_\lambda(u)x \, du \]
\[ = B_\lambda(t)F_\lambda(t)x \]

For all \( x \in D(A) \) we have

\[ F_\lambda(t)(\lambda - A)x = \int_0^t e^{-\lambda s}B_\lambda(s)(\lambda - A)x \, ds \]
\[ = \int_0^t e^{-\lambda s}(e^{\lambda s} - T(s))x \, ds \]
\[ = tx - \int_0^t e^{-\lambda s}T(s)x \, ds \]
\[ = tx - G_\lambda(t)B_\lambda(t)x \]
\[ = (\lambda - A)F_\lambda(t)x \]

\( \square \)

**Lemma 2.2.** Let \((T(t))_{t \geq 0}\) a \(C_0\)-semigroup on \(X\) with infinitesimal generator \(A\). For all \(\lambda \in \mathbb{C}\), \(t > 0\) and \(n \in \mathbb{N}\), there exists \(H_n(t), L_n(t) \in \mathcal{B}(X)\) such that

1. \(\forall x \in X, H_n(t)x \in D(A^n)\) and \((\lambda - A)^nH_n(t) + L_n(t)B_\lambda^n(t) = I\),

2. The operators \((\lambda - A)^n, H_n(t), L_n(t)\) and \(B_\lambda^n(t)\) are mutually commuting.

**Proof:** According to lemma 1 there exists tow bounded operators \(F_\lambda(t)\) and \(G_\lambda(t)\) such that \((\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = I\). For \(i \in \{1, \ldots, n - 1\}\) and \(x \in X\), we have

\[ (\lambda - A)^iF_\lambda^n(t)x = [(\lambda - A)F_\lambda(t)]^iF_\lambda^{n-i}(t)x \]
\[ = [F_\lambda(t)(\lambda - A)]^iF_\lambda^{n-i}(t)x \in D(A). \]

Hence \(\forall n \in \mathbb{N}^*, F_\lambda^n(t)x \in D(A^n)\). Therefore

\[ (\lambda - A)^nF_\lambda^n(t) = [(\lambda - A)F_\lambda(t)]^n \]
\[ = [I - G_\lambda(t)B_\lambda(t)]^n \]
\[ = I - L_{1,n}(t)B_\lambda(t) \]

with \(L_{1,n}(t) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} G_\lambda^k(t)B_\lambda^{k-1}(t)\). Hence \((\lambda - A)^n F_\lambda^n(t) + L_{1,n}(t)B_\lambda(t) = I\)
Similarly
\[ L_n^\lambda(t)B_n^\lambda(t) = [I - (\lambda - A)^n F_n^\lambda(t)]^n \]
\[ = I - (\lambda - A)^n \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (\lambda - A)^{n(k-1)} F_n^{\lambda k}(t) \]

Let \( H_n(t) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (\lambda - A)^{n(k-1)} F_n^{\lambda k}(t) \) and \( L_n(t) = L_{1,n}(t) \), then \((\lambda - A)^n H_n(t) + L_n(t)B_n^\lambda(t) = I \), moreover \((\lambda - A)^n \), \( H_n(t) \), \( L_n(t) \) and \( B_n^\lambda(t) \) are mutually commuting.

\[ \square \]

**Lemma 2.3.** Let \((T(t))_{t \geq 0}\) a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). If \( R(\lambda^\lambda - T(t))^p \) is closed, then \( R(\lambda - A)^p \) is closed.

**Proof:** Suppose that \( R(\lambda^\lambda - T(t))^p \) is closed. Let \( y_n = (\lambda - A)^p x_n \) be a convergent sequence with limit \( y \in X \). From lemma 2, there exists \( H_p(t), L_p(t) \in \mathcal{B}(X) \) such that \((\lambda - A)^p H_p(t) + L_p(t)B_n^\lambda(t) = I \), then \( x_n = (\lambda - A)^p H_p(t)x_n + L_p(t)B_n^\lambda(t)x_n \) and \( y_n = (\lambda - A)^p H_p(t)y_n + (\lambda^\lambda - T(t))^p L_p(t)x_n \). Since \((\lambda - A)^p H_p(t)\) is a linear bounded operator and \( R(\lambda^\lambda - T(t))^p \) is closed, then \((\lambda^\lambda - T(t))^p L_p(t)x_n = y_n - (\lambda - A)^p H_p(t)y \) tends to \( y - (\lambda - A)^p H_p(t)y \in R(\lambda^\lambda - T(t))^p \), therefore there exists \( z \in X \) such that \( y - (\lambda - A)^p H_p(t)y = (\lambda^\lambda - T(t))^p z \), then \( y = (\lambda - A)^p[H_p(t)y + B_n^\lambda(t)z] \), hence \( y \in R(\lambda - A)^p \).

\[ \square \]

We have the following theorem.

**Theorem 2.4.** Let \((T(t))_{t \geq 0}\) a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). Then

For all \( t \geq 0 \), \( e^{t\sigma_{\text{des}}(A)} \subseteq \sigma_{\text{des}}(T(t)) \setminus \{0\} \) and \( e^{t\sigma_{\text{asc}}(A)} \subseteq \sigma_{\text{asc}}(T(t)) \setminus \{0\} \)

**Proof:** If \( \lambda^\lambda - T(t) \) has finite descent, then there exists \( n \in \mathbb{N} \) such that \( R(\lambda^\lambda - T(t))^n = R(\lambda^\lambda - T(t))^{n+1} \), from lemma 3, there exist two operators \( H_n(t) \) and \( L_n(t) \) such that \((\lambda - A)^n H_n(t) + L_n(t)B_n^\lambda(t) = I \) and \( H_n(t), L_n(t), B_n^\lambda(t) \) and \( \lambda - A \) are mutually commuting. Let \( y \in R(\lambda - A)^n \) and \( x \in D(A^n) \) such that \( y = (\lambda - A)^nx \). Therefore

\[ (\lambda - A)^nx = (\lambda - A)^n H_n(t)(\lambda - A)^nx + L_n(t)B_n^\lambda(t)(\lambda - A)^nx \]
\[ = (\lambda - A)^{n+1} H_n(t)(\lambda - A)^{n-1}x + L_n(t)(\lambda^\lambda - T(t))^nx \]

Moreover, \( R(\lambda - A)^n = R(\lambda - A)^{n+1} \), hence \( \lambda - A \) has finite descent.

If \( \lambda^\lambda - T(t) \) has finite ascent, there exist \( n \in \mathbb{N} \) such that \( N(\lambda^\lambda - T(t))^n = N(\lambda^\lambda - T(t))^{n+1} \). Let \( x \in D(A)^{n+1} \), we have

\[ (\lambda - A)^nx = (\lambda - A)^n H_n(t)(\lambda - A)^nx + L_n(t)(\lambda^\lambda - T(t))^nx \]
\[ = (\lambda - A)^{n+1} H_n(t)(\lambda - A)^{n+1}x + L_n(t)(\lambda^\lambda - T(t))^nx \]

Moreover, \( N(\lambda - A)^n = N(\lambda - A)^{n+1} \), hence \( \lambda - A \) has finite ascent.

\[ \square \]
Remark 2.5. Consider the translation group on the space $C_{2\pi}(\mathbb{R})$ of all $2\pi$ periodic continuous functions on $\mathbb{R}$ and denote its generator by $A$ (see [5, Paragraph I.4.15]). From [5, Examples 2.6.iv] we have, $\sigma(A) = iz$, then $e^{\sigma(A)}$ is at most countable, therefore $e^{i\sigma_{asc}(A)}$ and $e^{i\sigma_{desc}(A)}$ are also. The spectra of the operators $T(t)$ are always contained in $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and contain the eigenvalues $e^{ikt}$ for $k \in \mathbb{Z}$. Since $\sigma(T(t))$ is closed, it follows from [5, Theorem IV.3.16] below, that $\sigma(T(t)) = \Gamma$ whenever $t/2\pi \notin \mathbb{Q}$, then $\sigma(T(t))$ is not countable, from [2, Corollary 2.10] and [3, Corollary 1.8], $\sigma_{asc}(T(t))$ and $\sigma_{desc}(T(t))$ are also. Therefore the inclusions of the preceding theorems are strict.

Corollary 2.6. Let $(T(t))_{t \geq 0}$ a $C_0$-semigroup on $X$ with infinitesimal generator $A$. Then

$$\text{For all } t \geq 0, \quad e^{t\sigma_D(A)} \subseteq \sigma_D(T(t)) \setminus \{0\}$$

Proof: If $e^{\lambda t} - T(t)$ is invertible Drazin, then $e^{\lambda t} - T(t)$ has finite ascent and descent $p$, therefore $R(e^{\lambda t} - T(t))^p$ is closed. By lemma 3 and theorem 1, $\lambda - A$ is invertible Drazin.

\[ \square \]

Remark 2.7. The inclusion of the preceding corollary is strict. Indeed, from remark 1, $e^{t\sigma_D(A)}$ is at most countable, on the other hand $\sigma_D(T(t))$ is not countable.

Theorem 2.8. Let $(T(t))_{t \geq 0}$ be an eventually norm-continuous semigroup with generator $A$ on the Banach space $X$. The spectral mapping theorem

$$e^{t\sigma_D(A)} = \sigma_D(T(t)) \setminus \{0\} \quad \text{for all } t \geq 0$$

holds.

Proof: Let $\lambda$ be a complex number such that $\lambda - A$ has finite ascent and descent $p$ such that $R(\lambda - A)^p$ is closed. According to [8, Lemma 3.4] and [8, Lemma 3.5], there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, the operator $\mu - A$ is bijective, by [9, Corollary 3.3], for every $\mu \in \mathbb{C}$ with $0 < |\mu - \lambda| < \delta$, $e^{\mu t} - T(t)$ is bijective, from open mapping theorem $e^{\lambda t}$ is isolated in $\sigma(T(t))$. By [1, Theorem 3.81], we have $e^{\lambda t}$ is a pole of the resolvent of $T(t)$. Using [7, Theorem V.10.1], we obtain $e^{\lambda t} - T(t)$ has a finite ascent and descent, moreover $e^{\lambda t} - T(t)$ is Drazin invertible.

\[ \square \]

Example 2.9. On $X := C_0(\Omega)$ take the multiplication operator $M_gf(\lambda) = q(\lambda)f(\lambda)$ for $\lambda \in \Omega$, $f \in X$. From [5, Proposition I.4.2] we obtain that $\sigma(M) = q(\Omega)$ and $\sigma_p(M) = \{\lambda \in \mathbb{C} : \lambda$ is isolated in $\Omega\}$. On for some continuous function $q : \Omega \rightarrow \mathbb{C}$, if $\sup_{s \in \Omega} Re(q(s)) < \infty$, then $T_q(t) := e^{t\lambda}$ defines a strongly continuous semigroup (see [5, Proposition I.4.5]). Suppose that $\Omega = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 1$ and $-1 \leq \text{Im}(\lambda) \leq 1\}$ and for all $\lambda \in \Omega$, $q(\lambda) = \lambda$. Then $\sigma(M) = \Omega$ and $\sigma_p(M) = \emptyset$, by [7, Theorem 5.41-C], we have $\sigma(M) = \sigma_{desc}(M) \cup \sigma_p(M) = \sigma_{desc}(M)$, then $\sigma_D(M) = \Omega$. Furthermore $q(\Omega) \cap \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \geq b\}$ is bounded for every $b \in \mathbb{R}$, from [5, Theorem II.4.32], $(T_q(t))_{t \geq 0}$ is eventually norm-continuous. By theorem 2, for $t > 0$, we have $\sigma_D(T(t)) = \{e^{it\lambda} : \lambda \in \Omega\} \cup \{0\}$.
Corollary 2.10. The spectral mapping theorem
\[ e^{t\sigma_D(A)} = \sigma_D(T(t)) \setminus \{0\} \] for all \( t \geq 0 \)
hold for the following classes of strongly continuous semigroups:
1. eventually compact semigroups,
2. eventually differentiable semigroups,
3. analytic semigroups.

Proof: If a strongly continuous semigroup \( (T(t))_{t \geq 0} \) satisfies one of the following conditions:
1. eventually compact semigroups,
2. eventually differentiable semigroups,
3. analytic semigroups.

Then it is an eventually norm-continuous semigroup, from Theorem 2 we have
\[ e^{t\sigma_D(A)} = \sigma_D(T(t)) \setminus \{0\} \] for all \( t \geq 0 \).

References

Abdelaziz Tajmouati and Hamid Boua,
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Laboratory of Mathematical Analysis and Applications,
Fes, Morocco.
E-mail address: abdelaziz.tajmouati@usmba.ac.ma hamid.boua@usmba.ac.ma