A New Differential Operator of Analytic Functions Involving Binomial Series

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ABSTRACT: In this paper, we introduce a new differential operator of analytic functions involving binomial series. Furthermore, we derive some subordination and superordination results for this operator. Some applications and examples are also obtained.

Key Words: Analytic functions, Differential subordinations, Differential superordinations, Dominant, Subordinant.

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1. Introduction and Definitions

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{U} := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If $p$ and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if $p$ satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \quad (1.2)$$

then $p$ is a solution of the differential superordination (1.2). (If $f$ is subordinate to $F$, then $F$ is superordinate to $f$.) An analytic function $q$ is called a subordinant if $q \prec p$ for all $p$ satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (1.2) is said to be the best subordinant. Miller and Mocanu [7] obtained conditions on $h$, $q$ and $\phi$ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

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Using the results of Miller and Mocanu [7], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [3] (see also, [1,5,10]). Shanmugam et al. [9] obtained sufficient conditions for a normalized analytic functions \( f(z) \) to satisfy

\[
q_1(z) \prec \frac{f(z)}{zf'(z)} < q_2(z) \quad \text{and} \quad q_1(z) < \frac{z^2f'(z)}{f(z)} < q_2(z).
\]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \).

For a function \( f \) in \( A \), and making use of the binomial series

\[
(1 - \lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j \quad (m \in \mathbb{N} = \{1, 2, \ldots\}, \ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),
\]

we now define the differential operator \( D^\zeta_{m,\lambda} f(z) \) as follows:

\[
D^0 f(z) = f(z), \quad (1.3)
\]

\[
D^1_{m,\lambda} f(z) = (1 - \lambda)^m f(z) + (1 - (1 - \lambda)^m)zf'(z) \quad (m \in \mathbb{N}, \ \lambda > 0), \quad (1.4)
\]

\[
D^\zeta_{m,\lambda} f(z) = D_{m,\lambda} (D^{\zeta-1}_{m,\lambda} f(z)) \quad (\zeta \in \mathbb{N}). \quad (1.5)
\]

If \( f \) is given by (1.1), then from (1.5) and (1.6) we see that

\[
D^\zeta_{m,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right) a_n z^n, \quad \zeta \in \mathbb{N}_0. \quad (1.7)
\]

Using the relation (1.7), it is easily verified that

\[
C^m_j(\lambda)z(D^\zeta_{m,\lambda} f(z))' = D^{\zeta+1}_{m,\lambda} f(z) - (1 - C^m_j(\lambda))D^\zeta_{m,\lambda} f(z) \quad (1.8)
\]

where \( C^m_j(\lambda) := \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \).

We observe that for \( m = 1 \), we obtain the differential operator \( D^\zeta_{1,\lambda} \) defined by Al-Oboudi [2] and for \( m = \lambda = 1 \), we get Sălăgean differential operator \( D^\zeta \) [8].

The main object of the present paper is to apply a method based on the differential subordination in order to derive several subordination results involving the operator \( D^\zeta_{m,\lambda} \). Furthermore, we obtain the previous results of Srivastava and Lashin [11] as special cases of some of the results presented here.

2. Preliminaries

In order to prove our results, we shall require the following known definition and lemmas.
Definition 2.1. [7, Definition 2, p. 817] Denote by $Q$, the set of all functions $f(z)$ that are analytic and injective on $U - E(f)$, where

$$E(f) = \{ \eta \in \partial U : \lim_{z \to \eta} f(z) = \infty \},$$

and are such that $f'(\eta) \neq 0$ for $\eta \in \partial U - E(f)$.

Lemma 2.2. [6, Theorem 3.4h, p. 132] Let $q(z)$ be univalent in the unit disk $U$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z)); h(z) = \theta(q(z)) + Q(z)$. Suppose that

1. $Q(z)$ is starlike univalent in $U$, and
2. $\Re \frac{zh'(z)}{q(z)} > 0$ for $z \in U$.

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3. [4] Let $q(z)$ be convex univalent in the unit disk $U$ and $\varphi$ and $\varphi$ be analytic in a domain $D$ containing $q(U)$. Suppose that

1. $\Re \left( \frac{\varphi'(q(z))}{\varphi(q(z))} \right) > 0$ for $z \in U$,
2. $zq'(z)\varphi(q(z))$ is starlike univalent in $U$.

If $p(z) \in K[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $U$, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Subordination for Analytic Functions

We begin by proving the following result.

Lemma 3.1. Let the functions $p(z)$ and $q(z)$ be analytic in $U$ and suppose that $q(z) \neq 0$ ($z \in U$) is also univalent in $U$ and that

$$\frac{zq'(z)}{q(z)}$$

is starlike univalent in $U$.

If $q(z)$ satisfies

$$\Re \left( 1 + \frac{c_1}{\beta} q(z) + \frac{2c_2}{\beta^2} (q(z))^2 + \cdots + \frac{nc_n}{\beta^n} (q(z))^n - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right) > 0$$

(3.2)
(z ∈ U; c₀, c₁, c₂, . . . , cₙ, β ∈ C; β ≠ 0)

and

\[ c₀ + c₁p(z) + c₂(p(z))^2 + \cdots + cₙ(p(z))^n + \beta \frac{zp'(z)}{p(z)} < c₀ + c₁q(z) + c₂(q(z))^2 + \cdots + cₙ(q(z))^n + \beta \frac{zq'(z)}{q(z)} \]

(z ∈ U; c₀, c₁, c₂, . . . , cₙ, β ∈ C; β ≠ 0)

then \( p(z) \prec q(z) \) (z ∈ U) and q is the best dominant.

**Proof:** Let

\[ \theta(\omega) := c₀ + c₁\omega + c₂\omega^2 + \cdots + cₙ\omega^n \text{ and } \phi(\omega) := \frac{\beta}{\omega}. \]

Then, we observe that \( \theta(\omega) \) is analytic in \( C \), \( \phi(\omega) \) is analytic in \( C^* = C \setminus \{0\} \) and that \( \phi(\omega) \neq 0 \) (\( \omega \in C^* \)).

Also, by letting

\[ Q(z) = zq'(z)\phi(q(z)) = \beta \frac{zq'(z)}{q(z)} \]

and

\[ h(z) = \theta(q(z)) + Q(z) = c₀ + c₁q(z) + c₂(q(z))^2 + \cdots + cₙ(q(z))^n + \beta \frac{zq'(z)}{q(z)} \]

we find from (3.1) and (3.2), \( Q(z) \) is starlike univalent in \( U \) and that

\[ \Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( 1 + \frac{a₁}{\beta} q(z) + \frac{2a₂}{\beta} (q(z))^2 + \cdots + \frac{naₙ}{\beta} (q(z))^n - \frac{zq'(z)}{q(z)} - \frac{zq''(z)}{q'(z)} \right) > 0 \]

(z ∈ U; c₀, c₁, c₂, . . . , cₙ, β ∈ C; β ≠ 0).

Our result now follows by an application of Lemma 2.2.

We first prove the following subordination theorem involving the operator \( D_{m,λ}^\zeta \).

**Theorem 3.2.** Let the function \( q(z) \) be analytic and univalent in \( U \) such that \( q(z) \neq 0 \) (z ∈ U). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \) and the inequality (3.2) holds true. Let

\[ \Omega_m^\zeta(c₀, c₁, c₂, . . . , cₙ, \zeta, \lambda, f) := c₀ + c₁ \left( \frac{D_{m,λ}^\zeta f(z)}{z} \right) + c₂ \left( \frac{D_{m,λ}^\zeta f(z)}{z} \right)^2 \]
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\[ + \cdots + c_n \left( \frac{D^\zeta_{m,\lambda} f(z)}{z} \right)^n + \frac{\beta}{C^m_j (\lambda)} \left( \frac{D^{\zeta+1}_{m,\lambda} f(z)}{D^{\zeta}_{m,\lambda} f(z)} - (1 - C^m_j (\lambda)) \right). \] (3.3)

If \( q(z) \) satisfies

\[ \Omega^m_j (c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, f) \prec c_0 + c_1 q(z) + c_2 (q(z))^2 + \cdots + c_n (q(z))^n + \beta \frac{z q'(z)}{q(z)} \] (3.4)

\[ (z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0) \]

then

\[ \frac{D^\zeta_{m,\lambda} f(z)}{z} \prec q(z) \quad (z \in \mathbb{U}\{0\}) \]

and \( q \) is the best dominant.

**Proof:** Define the function \( p(z) \) by

\[ p(z) := \frac{D^\zeta_{m,\lambda} f(z)}{z} \quad (z \in \mathbb{U}\{0\}; f \in A). \]

Then a computation shows that

\[ \frac{z p'(z)}{p(z)} = \frac{z (D^\zeta_{m,\lambda} f(z))'}{D^\zeta_{m,\lambda} f(z)} - 1. \]

By using the identity (1.8), we obtain

\[ \frac{z p'(z)}{p(z)} = \frac{1}{C^m_j (\lambda)} \left( \frac{D^{\zeta+1}_{m,\lambda} f(z)}{D^{\zeta}_{m,\lambda} f(z)} - (1 - C^m_j (\lambda)) \right) \]

which, in light the hypothesis (3.4), yields the following subordination

\[ c_0 + c_1 p(z) + c_2 (p(z))^2 + \cdots + c_n (p(z))^n + \beta \frac{z p'(z)}{p(z)} \prec c_0 + c_1 q(z) + c_2 (q(z))^2 + \cdots + c_n (q(z))^n + \beta \frac{z q'(z)}{q(z)} \]

and Theorem 3.2 follows by an application of Lemma 3.1.

For the choices \( q(z) = \frac{1 + \lambda z}{1 + B z} \), \(-1 \leq B < A \leq 1\) and \( q(z) = \left( \frac{1 + 2z}{1 + 2z} \right)^\mu \), \(0 < \mu \leq 1\) in Theorem 3.2, we get Corollaries 3.3 and 3.4 below.
Corollary 3.3. Assume that (3.2) holds true. If \( f \in \mathcal{A} \) and
\[
\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \prec c_0 + c_1 \left(1 + \frac{A_z}{1 + B_z}\right) + c_2 \left(1 + \frac{A_z}{1 + B_z}\right)^2 + \cdots + c_n \left(1 + \frac{A_z}{1 + B_z}\right)^n + \frac{\beta(A - B)z}{(1 + A_z)(1 + B_z)}
\]
\((z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0)\),
where \( \Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \) is as defined in equation (3.3), then
\[
D_{z}^{\zeta} \prec \frac{1 + A_z}{1 + B_z} (z \in \mathbb{U}\{0\})
\]
and \( \frac{1 + A_z}{1 + B_z} \) is the best dominant.

Corollary 3.4. Assume that (3.2) holds true. If \( f \in \mathcal{A} \) and
\[
\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \prec c_0 + c_1 \left(1 + \frac{A_z}{1 - z}\right)^\mu + c_2 \left(1 + \frac{z}{1 - z}\right)^{2\mu} + \cdots + c_n \left(1 + \frac{z}{1 - z}\right)^{2n\mu} + 2\beta \frac{z}{1 - z^2}
\]
\((z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0)\),
where \( \Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \) is as defined in equation (3.3), then
\[
D_{z}^{\zeta} \prec \left(1 + \frac{z}{1 - z}\right)^\mu (z \in \mathbb{U}\{0\})
\]
and \( \left(1 + \frac{z}{1 - z}\right)^\mu \) is the best dominant.

For \( q(z) = e^{A_z}; (|\epsilon A| < \pi) \), in Theorem 3.2, we get the following result.

Corollary 3.5. Assume that (3.2) holds true. If \( f \in \mathcal{A} \) and
\[
\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \prec c_0 + c_1 e^{\epsilon A_z} + c_2 e^{2\epsilon A_z} + \cdots + c_n e^{n\epsilon A_z} + \beta \epsilon A_z
\]
\((z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0)\),
where \( \Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \) is as defined in equation (3.3), then
\[
D_{z}^{\zeta} \prec e^{A_z} (z \in \mathbb{U}\{0\})
\]
and $e^{\beta Az}$ is the best dominant.

For $q(z) = \frac{1}{(1-z)^{2b}}$, $b \in \mathbb{C}^*$, $c_0 = \zeta = \lambda = m = 1, c_1 = c_2 = \ldots = c_n = 0$ and $\beta = \frac{1}{b}$ in Theorem 3.2, we get the following result obtained by Srivastava and Lashin [11].

**Corollary 3.6.** Let $b$ be a non zero complex number. If $f \in A$, and

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] < \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} < \frac{1}{(1-z)^{2b}},$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

### 4. Superordination for Analytic Functions

Next, applying Lemma 2.3, we obtain the following two theorems.

**Theorem 4.1.** Let $q$ be analytic and convex univalent in $U$ such that $q(z) \neq 0$ and $\frac{z q'(z)}{q(z)}$ is starlike univalent in $U$. Suppose also that

$$\Re\left(\frac{c_1 q(z)}{\beta} + \frac{2c_2}{\beta}(q(z))^2 + \ldots + \frac{nc_n}{\beta}(q(z))^n\right) > 0 \quad (4.1)$$

$$(z \in U; c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0).$$

If $f \in A$,

$$D_{m,\lambda}^\zeta f(z) \in \mathcal{H}[q(0), 1] \cap Q$$

and $\Omega^\mu_m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda)$ defined in $(3.3)$ is univalent in $U$, then the following superordination:

$$c_0 + c_1 q(z) + c_2 (q(z))^2 + \ldots + c_n(q(z))^n + \beta \frac{z q'(z)}{q(z)} \prec \Omega^\mu_m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$$

$$(z \in U; c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0).$$

implies that

$$q(z) \prec \frac{D_{m,\lambda}^\zeta f(z)}{z} \quad (z \in U \setminus \{0\})$$

and $q(z)$ is the best subordinant.

**Proof:** Let

$$\vartheta(\omega) := c_0 + c_1 \omega + c_2 \omega^2 + \ldots + c_n \omega^n \text{ and } \varphi(\omega) := \beta \frac{\omega'}{\omega}. $$
Then, we observe that \( \vartheta(\omega) \) is analytic in \( \mathbb{C} \), \( \varphi(\omega) \) is analytic in \( \mathbb{C}^* = \mathbb{C}\setminus\{0\} \) and that \( \varphi(\omega) \neq 0 \) (\( \omega \in \mathbb{C}^* \)).

Since \( q \) is a convex univalent in \( \mathbb{U} \), it follows that

\[
\Re\left( \frac{\vartheta'(q(z))}{\varphi(q(z))} \right) = \Re\left( \frac{c_1}{\beta} q(z) + \frac{2c_2}{\beta} (q(z))^2 + \cdots + \frac{n c_n}{\beta} (q(z))^n \right) > 0
\]

(\( z \in \mathbb{U}; c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0 \)).

Theorem 4.1 follows as an application of Lemma 2.3. \( \square \)

Combining the results of differential subordination and superordination, we state the following "sandwich results":

**Theorem 4.2.** Let \( q_1 \) be convex univalent and \( q_2 \) be univalent in \( \mathbb{U} \) such that \( q_1(z) \neq 0 \) and \( q_2(z) \neq 0 \) (\( z \in \mathbb{U} \)). Suppose also that \( q_2 \) satisfies (4.1) and \( q_1 \) satisfies (3.2). If \( f \in \mathcal{A} \),

\[
D^\mathcal{K}_{m,\lambda} f(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}
\]

and

\[
c_0 + c_1 \left( \frac{D^\mathcal{K}_{m,\lambda} f(z)}{z} \right) + c_2 \left( \frac{D^\mathcal{K}_{m,\lambda} f(z)}{z} \right)^2 + \cdots + c_n \left( \frac{D^\mathcal{K}_{m,\lambda} f(z)}{z} \right)^n + \frac{\beta}{C_j^m(\lambda)} \left( \frac{D^\mathcal{K}_{m,\lambda} f(z)}{D^\mathcal{K}_{m,\lambda} f(z)} - (1 - C_j^m(\lambda)) \right)
\]

(\( z \in \mathbb{U}; c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0 \)).

is univalent in \( \mathbb{U} \), then the subordination given by

\[
c_0 + c_1 q_1(z) + c_2 (q_1(z))^2 + \cdots + c_n (q_1(z))^n + \frac{\beta q_1(z)}{q_1(z)} - \Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) < 0
\]

\[
< c_0 + c_1 q_2(z) + c_2 (q_2(z))^2 + \cdots + c_n (q_2(z))^n + \frac{\beta q_2(z)}{q_2(z)}
\]

(\( z \in \mathbb{U}; c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0 \)).

implies that

\[
q_1(z) \prec \frac{D^\mathcal{K}_{m,\lambda} f(z)}{z} \prec q_2(z).
\]

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References


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