On Convex Hull of Orthogonal Scalar Spectral Functions of a Carleman Operator

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ABSTRACT: In this paper we describe the closed convex hull of orthogonal resolvents of an abstract symmetric operator of defect indices (1,1), then we study the convex hull of orthogonal spectral functions of a Carleman operator in the Hilbert space $L^2(X,\mu)$.

Key Words: defect indices, integral operator, spectral theory.

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1. Introduction

Let $H$ be a (separable) Hilbert space and let $A$ be a symmetric operator in $H$ with defect indices $(1,1)$. It is well known that the set $W(A)$ of all generalized spectral functions of $A$ is both convex and closed (in some natural topology). Consider the following problem:

Describe a convex hull $W_0(A)$ of orthogonal spectral functions of $A$.

This problem has been solved by I.M. Glazman [7] for Jacobi matrices corresponding to Hamburger moment problem. To explain his result let us recall that according to the Krein-Naimark formula for generalized resolvents of $A$ the set $W(A)$ is described as follows: $E_t \in W(A)$ if and only if

$$
\int \frac{d(E_t f; f)}{t-\lambda} = \frac{D_0(\lambda) \varphi(\lambda) + D_1(\lambda)}{C_0(\lambda) \varphi(\lambda) + C_1(\lambda)}, \quad \varphi \in N,
$$

where $D_i, C_i$ ($i=1,2$) are entries of the resolvent matrix of $A$ and $f$ is a "scale" vector and $N$ stands for the Nevanlinna class of functions holomorphic in $\mathbb{C}_+$ with non-negative imaginary parts [9].

Glazman [7] proved that $E_t \in W_0(A)$ if and only if $\varphi$ in (1) admits a representation

$$
\varphi(\lambda) = \varphi^*(C_1(\lambda)/C_0(\lambda)), \quad \varphi^* \in N.
$$

Though Glazman proved this result for Jacobi matrices, it remains valid for any symmetric operator $A$ with defect indices $(1,1)$. This result may be found in

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However, the orthogonal resolvents and the orthogonal spectral functions, which are those of the selfadjoint extensions in the same space, play a central role. We start by describing the closed convex hull of orthogonal resolvents of an abstract symmetric operator of defect indices \((1, 1)\), then we study the convex hull of orthogonal spectral functions of a Carleman operator in the Hilbert space \(L^2 (X, \mu)\). Let us notice that the same problem for the differential operator of second order on \((0, \infty)\) was considered in \([8]\).

2. Resolvents of a symmetric operator of defect indices \((1, 1)\)

We begin by introducing the following sets: 
\(\Phi\) the set of the analytical functions \(\varphi (z)\) on the unit disc \(K = \{ z \in \mathbb{C} : |z| < 1 \}\) such that \(|\varphi (z)| \leq 1, z \in K\); 
\(\mathcal{M}\) the set of all functions \(\varphi (z) \in \Phi\) admitting the representation

\[
\varphi (z) = \frac{\int_0^{2\pi} \frac{e^{it}}{1-ze^{it}} dS (t)}{\int_0^{2\pi} \frac{1}{1-ze^{it}} dS (t)}
\]

where \(S (t)\) is a monotonic nondecreasing function with total variation equal to one, i.e. \(\int_0^{2\pi} dS (t) = 1\); 
\(\mathcal{M}_0\) the set of all functions \(\varphi (z) \in \mathcal{M}\) with \(S (t)\) a step function with a finite number of jumps.

Lemma 2.1. 1. \(\mathcal{M}\) is closed under pointwise convergence and \(\mathcal{M}_0\) is dense in \(\mathcal{M}\). 
2. \(\mathcal{M} = \Phi\).

Proof:
1. We assume that for all \(z \in K\)

\[
\lim_{n \to \infty} \varphi_n (z) = \varphi (z),
\]

with

\[
\varphi_n (z) = \frac{\int_0^{2\pi} \frac{e^{it}}{1-ze^{it}} dS_n (t)}{\int_0^{2\pi} \frac{1}{1-ze^{it}} dS_n (t)} \in \mathcal{M}.
\]

According to a theorem by Helly \([10]\), there exists a nondecreasing function \(S (t)\) with total variation equal to 1 and a subsequence \(n_k\) such that

\[
\lim_{n_k \to \infty} S_{n_k} (t) = S (t)
\]

in each point of continuity of \(S (t)\). Thus we have

\[
\lim_{n_k \to \infty} \varphi_{n_k} (z) = \varphi (z) = \frac{\int_0^{2\pi} \frac{e^{it}}{1-ze^{it}} dS (t)}{\int_0^{2\pi} \frac{1}{1-ze^{it}} dS (t)} \in \mathcal{M},
\]

and the density of \(\mathcal{M}_0\) in \(\mathcal{M}\) is then straightforward.
2. Now, we introduce two other sets:

\( \Phi^+ \) the set of all functions \( f(\lambda) \) analytic in the upper half-plane \( \Pi^+ = \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \} \) with positive imaginary part, i.e. \( \Im \lambda \geq 0 \), and \( M^+ \) the set of functions \( \tau(\lambda) \in \Phi^+ \) having the following representation

\[
\tau(\lambda) = -\left( \lambda + \frac{1}{F(\lambda)} \right),
\]

with

\[
F(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - \lambda},
\]

\( \mu(t) \) being a nondecreasing function with total variation equal to one.

It is easy to see that the homographic transformation

\[
L(\lambda) = \frac{\lambda - i}{\lambda + i} = z
\]

establishes a bijection between the sets \( \Phi \) and \( \Phi^+ \). Indeed if, \( f(\lambda) \in \Phi^+ \), then

\[
\varphi(z) = L \left( f \left( L^{-1}(z) \right) \right) = L \left( f \left( \frac{1 + i}{1 + z} \right) \right) = \frac{f \left( \frac{1 + i}{1 + z} \right) - i}{f \left( \frac{1 + i}{1 + z} \right) + i} \in \Phi
\]

and, reciprocally, if \( \varphi(z) \in \Phi \), then

\[
f(\lambda) = L^{-1} \left( \varphi(L(\lambda)) \right) \in \Phi^+.
\]

We can also see that (4) establishes the same bijection enters \( M \) and \( M^+ \). Indeed, we have for \( \tau(\lambda) \in M^+ \)

\[
-\left( \lambda + \frac{1}{F(\lambda)} \right) = -\left( \lambda + \frac{1}{\int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - \lambda}} \right) = \frac{\int_{-\infty}^{+\infty} \frac{t d\mu(t)}{t - \lambda}}{\int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - \lambda}}. \tag{5}
\]

And making in the formula (5) the following change of variables

\[
\frac{\lambda - i}{\lambda + i} = z; \quad \lambda = i \frac{1 + z}{1 - z}; \quad e^{i\alpha} = \frac{t - i}{t + i}; \quad t = i \frac{1 + e^{i\alpha}}{1 - e^{i\alpha}} = -\cot \frac{\alpha}{2},
\]

we will have

\[
-\left( \lambda + \frac{1}{F(\lambda)} \right) = -i \int_{0}^{2\pi} \frac{1 + e^{i\alpha}}{e^{i\alpha} - z} dS(\alpha) = f(z),
\]

where \( S(\alpha) = \mu \left( -\cot \frac{\alpha}{2} \right) \). Thus

\[
\frac{f(z) - i}{f(z) + i} = \frac{i \int_{0}^{2\pi} \frac{1}{e^{i\alpha} - z} dS(\alpha)}{\int_{0}^{2\pi} \frac{1}{e^{i\alpha} - z} dS(\alpha)} \in M.
\]

But we know [8] that \( M^+ \) is dense in \( \Phi^+ \), consequently \( M \) is dense in \( \Phi \) for pointwise convergence. As \( M \) is closed for this convergence then \( M = \Phi \).
Let $A$ be a closed symmetric operator defined in a Hilbert space $H$ with the scalar product $(.,.)$. Its domain of definition $D(A)$ is assumed to be dense in $H$. For $\lambda \in \mathbb{C}$, $\exists \lambda \neq 0$ we set
\[
\mathfrak{M}_\lambda = (A - \lambda I) D(A), \quad \mathfrak{N}_\lambda = H \cap \mathfrak{M}_\lambda;
\]
where $I$ is the identity operator in $H$. We recall that $D(A)$ and the defect subspaces $\mathfrak{M}_\lambda$ and $\mathfrak{N}_\lambda$ are linearly independent.

In the sequel we suppose the operator $A$ with defect indices $(1,1)$, i.e.
\[
\dim \mathfrak{M}_\lambda = \dim \mathfrak{N}_\lambda = 1.
\]

We now state the formula of generalized resolvents $R_\omega(\lambda)$ of the operator $A$ obtained in [2] and [4]
\[
R_\omega(\lambda) f = (A_\omega - \lambda I)^{-1} f = \tilde{R}(\lambda) f + \frac{1 - \omega(\lambda)}{\omega(\lambda) C(\lambda) - 1} \cdot \frac{(f, \varphi_\lambda)}{(\lambda + i)(\varphi_\lambda, \varphi_i)} \varphi_\lambda, \quad (6)
\]
where $\tilde{R}(\lambda) = \left(\tilde{A} - \lambda I\right)^{-1}$ is the resolvent of the selfadjoint extension $\tilde{A}$ of the operator $A$, $\omega(\lambda)$ an analytical function in $\Pi^+$ such that $|\omega(\lambda)| \leq 1$, $\Im(\lambda) > 0$, $\varphi_i \in \mathfrak{N}_{-i}$, $\varphi_\lambda = \tilde{U}_\lambda \varphi_i = \left(\tilde{A} - iI\right) \left(\tilde{A} - \lambda I\right)^{-1} \varphi_i \in \mathfrak{N}_{\lambda}$ [9] and
\[
C(\lambda) = \frac{\lambda - i}{\lambda + i} \frac{(\varphi_\lambda, \varphi_{-i})}{(\varphi_\lambda, \varphi_i)}
\]
the characteristic function of $A$ checking for $\lambda, \Im \lambda > 0 : |C(\lambda)| < 1$.

The generalized resolvent defined by (6) is a resolvent of a selfadjoint extension of $A$ if and only if $\omega(\lambda) = \varkappa = \text{const}$, $|\varkappa| = 1$. We will call it orthogonal ( or canonical ) resolvent.

Every generalized resolvent $R_\omega(\lambda)$ is connected with the generalized spectral function $E_\omega^\varkappa$ by the relation
\[
R_\omega(\lambda) = \int_{-\infty}^{+\infty} \frac{1}{t - \lambda} dE_\omega^\varkappa \quad (7)
\]
As the set of the generalized spectral functions is convex [2], then the set of generalized resolvent is also.

Let $\mathfrak{R}_0$ be the convex set of the orthogonal resolvents $R_{\varkappa_j}(\lambda)$ ($j = 1, 2, ..., n$), corresponding to the constants $\varkappa_j, |\varkappa_j| = 1$, ($j = 1, 2, ..., n$), i.e.
\[
\mathfrak{R}_0 = \left\{ R_\omega(\lambda) = \sum_{j=1}^{n} \alpha_j R_{\varkappa_j}(\lambda), \ \alpha_j > 0, \ \sum_{j=1}^{n} \alpha_j = 1 \right\}.
\]
Thus we have for an \( R (\lambda) \rightarrow R_0 \) and \( f \in H \)

\[
R_\omega (\lambda) f = \tilde{R} (\lambda) f - \sum_{j=1}^{n} \alpha_j \frac{1 - \nu_j}{1 - \nu_j C (\lambda)} h (\lambda) \inner{f, \varphi_j} \varphi_j
\]

\[
= \tilde{R} (\lambda) f - \frac{1 - \omega (\lambda)}{1 - \omega (\lambda) C (\lambda)} h (\lambda) \inner{f, \varphi},
\]

with

\[
h (\lambda) = [(\lambda + 1) (\varphi_\lambda, \varphi_i)]^{-1}.
\]

As

\[
\frac{1 - \omega (\lambda)}{1 - \omega (\lambda) C (\lambda)} = \sum_{j=1}^{n} \nu_j \frac{1 - \nu_j}{1 - \nu_j C (\lambda)},
\]

we obtain

\[
\omega (\lambda) = \sum_{j=1}^{n} \frac{\alpha_j \nu_j}{1 - \nu_j C (\lambda)} = \int_{0}^{2\pi} \frac{\varphi'^*}{1 - C (\lambda) e^{it}} dS (t) = \varphi (C (\lambda)),
\]

where \( \varphi (z) \in \mathcal{M}_0 \).

Let us denote by \( R = \overline{R}_0 \) the closed convex hull of orthogonal resolvents in strong topology.

**Theorem 2.1.** The closed convex hull \( R \) is characterized by the formula (6), with \( \omega (\lambda) = \varphi (C (\lambda)) \) and \( \varphi (z) \) is an arbitrary function from \( \mathcal{M} \).

**Proof:** If \( R_\omega (\lambda) \in \mathcal{R}_0 \), then \( \omega (\lambda) = \varphi (C (\lambda)) \), \( \varphi (z) \in \mathcal{M}_0 \). We assume that \( R_\omega (\lambda) \in \mathcal{R} \) and \( R_\omega (\lambda) \notin \mathcal{R}_0 \). Therefore, there is a sequence \( R_{\omega_n} (\lambda) \) convergent to \( R_\omega (\lambda) \), as \( n \to \infty \), \( R_{\omega_n} (\lambda) \in \mathcal{R}_0 \) for strong topology, i.e.

\[
R_{\omega_n} (\lambda) f = \tilde{R} (\lambda) f - \frac{1 - \omega_n (\lambda)}{1 - \omega_n (\lambda) C (\lambda)} h (\lambda) \inner{f, \varphi_j} \varphi_j,
\]

with \( \omega_n (\lambda) = \varphi_n C (\lambda), \varphi_n (z) \in \mathcal{M}_0 \).

As

\[
\lim_{n \to \infty} R_{\omega_n} (\lambda) = R_\omega (\lambda)
\]

it follows that

\[
\lim_{n \to \infty} \left[ 1 - \omega_n (\lambda) \right] = \left[ 1 - \omega (\lambda) \right]^{-1}
\]

for all \( \lambda, \exists \lambda > 0 \). Finally, according to the lemma

\[
\lim_{n \to \infty} \omega_n (\lambda) = \lim_{n \to \infty} \varphi_n (C (\lambda)) = \omega (\lambda) = \varphi (C (\lambda)),
\]
Let $E_{\lambda}^{\nu}$, $j = 1, 2, ..., n$ be the orthogonal spectral functions connected with the orthogonal resolvents $R_{\lambda,j}^{R \setminus \nu}$ by (7). We denote by $E_0$ the convex hull of orthogonal spectral functions $E_0 = \left\{ \begin{array}{lcl} E_j = \sum_{j=1}^{n} \alpha_j E_{\lambda}^{\nu}, & \alpha_j > 0, & \sum_{j=1}^{n} \alpha_j = 1 \end{array} \right\}$

and $E = E_0$ for the strong topology. It is obvious that each $E_{\lambda}^{\nu} \in E$ defines by (7) a generalized resolvent $R_{\lambda,j}^{R \setminus \nu}$ and, reciprocally, each generalized resolvent $R_{\lambda,j}^{R \setminus \nu} \in E$ defines by (7) a generalized spectral function $E_{\lambda}^{\nu} \in E$.

3. Orthogonal scalar spectral functions of a Carleman operator

Let $X$ be an arbitrary set, $\mu$ a $\sigma$-finite measure on $X$, $L_2 (X, \mu)$ the Hilbert space of square integrable functions defined on $X$, $\{\psi_p (x)\}_{p=1}^{\infty}$ an orthonormal sequence in $L_2 (X, \mu)$, $\{\gamma_p\}_{p=1}^{\infty}$ a sequence of real numbers such that $\sum_{p=1}^{\infty} \gamma_p \psi_p (x) = 0$ for almost all $x \in X$, $K (x, y)$ a Carleman kernel

$$K (x, y) = \sum_{p=0}^{\infty} a_p \psi_p (x) \overline{\psi_p (y)} ,$$

where $\{a_p\}_{p=1}^{\infty}$ is a sequence of real numbers.

We assume that for almost all $x \in X$

$$\sum_{p=0}^{\infty} \left| a_p \psi_p (x) \right|^2 < \infty , \quad \sum_{p=0}^{\infty} \left| \psi_p (x) \right|^2 < \infty$$

and

$$\sum_{p=0}^{\infty} \left| \frac{\gamma_p}{a_p - \lambda} \right|^2 < \infty .$$

With these conditions, the symmetric operator $A = (A^*)^*$ admits the defect indices (1, 1) \[3\],

$$A^* f (x) = \sum_{p=0}^{\infty} a_p (f, \psi_p) \psi_p (x) ,$$

$$D (A^*) = \left\{ f \in L^2 (X, \mu) : \sum_{p=0}^{\infty} a_p (f, \psi_p) \psi_p (x) \in L^2 (X, \mu) \right\} ,$$

and moreover, we have

$$\begin{array}{lcl} \varphi_{\lambda} (x) = \sum_{p=0}^{\infty} \frac{\gamma_p}{a_p - \lambda} \psi_p (x) \in \mathcal{H}, & \lambda \in \mathbb{C}, & \lambda \neq a_k, \ k = 1, 2, ... \\
\varphi_{a_k} (x) = \psi_k (x) . 
\end{array}$$
We denote by $L_\psi$ the sub-space of $L_2(X,\mu)$ generated by the sequence $\{\Psi_p(x)\}_{p=0}^\infty$. As $L_\psi$ is reduced by $A$, we consider $A$ on $L_\psi$. Then, we have for all $f \in L_\psi$:[4]

$$f(x) = \int_{-\infty}^{+\infty} \frac{(f,\varphi_\sigma)\varphi_\sigma(x)}{(\sigma^2+1)|\varphi_\sigma,\varphi_i|} d\rho(\sigma) \quad \text{for almost all } x \in X,$$

$$\|f\|^2 = \int_{-\infty}^{+\infty} \frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)|\varphi_\sigma,\varphi_i|^2} d\rho(\sigma), \quad (8)$$

with

$$\varphi_i^\circ = \frac{\varphi_i}{\|\varphi_i\|},$$

and

$$\rho(\sigma) = \frac{1}{\pi} \lim_{\tau \to 0} \int_0^\sigma \frac{\Im \frac{1+\omega(t+i\tau)C(t+i\tau)}{1-\omega(t+i\tau)C(t+i\tau)} dt}{}, \quad (9)$$

$\omega(\lambda)$ is an analytical function on $\Pi^+$ with $|\omega(\lambda)| \leq 1, (3\lambda > 0)$. We call the function $\rho(\sigma)$ scalar spectral function of the operator $A$.

Let us designate by $\Psi$, the set of the functions defined by (9) with $\omega(\lambda)$ analytic in $\Pi^+$ and $|\omega(\lambda)| \leq 1$. As the set of the spectral functions of a symmetric operator is convex thus $\Psi$ is too. Subsequently we will assume that $\rho(\sigma) \in \Psi$ is normalized by continuity on the left, i.e. $\rho(\sigma) = \rho(\sigma - 0)$. In $\Psi$, we consider pointwise convergence, i.e one says that $\rho_n(t) \to \rho(t), \rho_n(t) \in \Psi, n \to \infty$ if $\lim_{n \to \infty} \rho_n(t) = \rho(t)$ for each continuity point of $\rho(t)$.

**Theorem 3.1.** The set $\Psi$ is closed for pointwise convergence.

**Proof:** Let's $\rho_n(t) \in \Psi$ and $\rho_n(\sigma) \to \rho(\sigma), n \to \infty$. It is suffice to establish the possibility of the passage to the limit under the integral sign:

$$\|f\|^2 = \int_{-\infty}^{+\infty} \frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)|\varphi_\sigma,\varphi_i|^2} d\rho(\sigma), \quad f \in L_\psi$$

as $n$ tends to $\infty$. 

However, for all $f \in D(A)$ and for all $b > 0$

\[
\int_b^{+\infty} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma) = \int_b^{+\infty} \frac{1}{\sigma^2} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma)
\leq \frac{1}{b^2} \int_b^{+\infty} \frac{|(f, A^* \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma)
= \frac{1}{b^2} \int_b^{+\infty} \frac{|(Af, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma)
\leq \frac{1}{b^2} \int_b^{+\infty} \frac{|(Af, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma)
= \frac{1}{b^2} \|Af\| \to 0, \text{ as } b \to +\infty.
\]

In the same way, we show that :

\[
\int_{-\infty}^{-a} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma) \to 0 \text{ as } a \to +\infty.
\]

By applying the modified theorem of Helly [10], we have for all $f \in D(A)$

\[
\|f\|^2 = \int_{-\infty}^{+\infty} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma) \to \int_{-\infty}^{+\infty} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho_n(\sigma),
\]

as $n \to \infty$, or

\[
\|f\|^2 = \int_{-\infty}^{+\infty} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)\left|\left(\varphi_\sigma, \varphi_i^0\right)\right|^2} d\rho(\sigma)
\]

As $\overline{D(A)} = \mathcal{L}_\psi$, this equality is true for all $f \in \mathcal{L}_\psi$. \hfill \Box

We will call $\rho_{\kappa}(\sigma) = \rho(\sigma) \in \mathcal{P}$ orthogonal scalar spectral function if it corresponds to a constant $\kappa, |\kappa| = 1$.

Let’s $\rho_{\kappa_k}, |\kappa_k| = 1, 2, ..., n$ be orthogonal scalar spectral functions corresponding to the constants $\kappa_1, \kappa_2, ..., \kappa_n$.

We denote by $\mathcal{G}_0$ the convex hull of these functions :

\[
\mathcal{G}_0 = \left\{ \rho(\sigma) = \sum_{k=1}^{n} \alpha_k \rho_{\kappa_k}, \alpha_k > 0, \sum_{k=1}^{n} \alpha_k = 1 \right\}
\]

and $\mathcal{G} = \overline{\mathcal{G}_0}$ for the convergence in each point of continuity.
For any function $\rho(\sigma) \in \mathfrak{S}_0$ we have:

$$\rho(\sigma) = \frac{1}{\pi} \lim_{\tau \to -\infty} \int_0^\sigma \Re \left[ \sum_{k=1}^{n} \frac{\alpha_k}{1 - z_k \alpha_k C(t + i\tau)} \left( \frac{1 + z_k C(t + i\tau)}{1 - z_k C(t + i\tau)} \right) \right] dt = \frac{1}{\pi} \lim_{\tau \to -\infty} \int_0^\sigma \Re \left[ \frac{1 + \omega(t + i\tau) C(t + i\tau)}{1 - \omega(t + i\tau) C(t + i\tau)} \right] dt$$

(10)

where $\omega(\lambda)$ is the analytical function corresponding to $\rho(\sigma) \in \mathfrak{P}$. From where we finds easily:

$$\omega(\lambda) = \frac{\sum_{k=1}^{n} \frac{\alpha_k}{1 - z_k \alpha_k C(\lambda)}}{\sum_{j=1}^{n} \frac{1}{1 - z_j \alpha_j C(\lambda)}}$$

$$= \frac{\int_{-\infty}^{\infty} \frac{e^{it}}{1 - e^{it} \alpha C(\lambda)} dS(t)}{\int_{-\infty}^{\infty} \frac{1}{1 - C(\lambda) e^{it}} dS(t)} = \varphi(C(\lambda)).$$

with $\varphi(z) \in \mathfrak{M}_0$.

**Theorem 3.2.** The closed convex hull $\mathfrak{S}$ of orthogonal scalar spectral functions, is described by (9) where $\omega(\lambda)$ is of the form $\omega(\lambda) = \varphi(C(\lambda))$ and $\varphi(z)$ is an arbitrary function of $\mathfrak{M}$.

**Proof:** Let $\rho(t) \in \mathfrak{S}_0$, then according to (11) $\omega(\lambda)$ which corresponds to $\rho(t)$ has the form $\omega(\lambda) = \varphi(C(\lambda))$ with $\varphi(z) \in \mathfrak{M}_0$.

We suppose, now, that $\rho(t) \in \mathfrak{S}$ and $\rho(t) \notin \mathfrak{G}_0$, there is then a sequence $\rho_n(t) \in \mathfrak{S}_0$ which converges to $\rho(t)$ in each point of continuity of $\rho(t)$. Each function $\rho_n(t) \in \mathfrak{S}_0$ corresponds to a function $\omega_n(\lambda) = \varphi_n(C(\lambda))$ with $\varphi_n(z) \in \mathfrak{M}_0$. As

$$\Re \frac{1 + \varphi_n(C(\lambda)) C(t + i\tau)}{1 - \varphi_n(C(\lambda)) C(t + i\tau)} > 0 \quad (3\lambda > 0),$$

by applying the inversion formula of Stieltjes [1], we will have:

$$\int_{-\infty}^{+\infty} \frac{1}{t - \lambda} d\rho_n(t) = \frac{1 - \varphi_n(C(\lambda)) C(t + i\tau)}{1 + \varphi_n(C(\lambda)) C(t + i\tau)}$$

(12)

And while making tend $n$ to $\infty$, it is followed from there:

$$\int_{-\infty}^{+\infty} \frac{1}{t - \lambda} d\rho(t) = \frac{1 - \varphi(C(\lambda)) C(t + i\tau)}{1 + \varphi(C(\lambda)) C(t + i\tau)}$$

As $\varphi_n(z) \in \mathfrak{M}_0$ and $\mathfrak{M}_0 = \mathfrak{M}$ then $\varphi(z) \in \mathfrak{M}$.

Reciprocally, let $\omega(\lambda) = \varphi(C(\lambda))$ with $\varphi(z) \in \mathfrak{M}$, the function $\omega(\lambda)$ corresponds to a spectral function $\rho(t)$. Let us show that $\rho(t) \in \mathfrak{S}$. Let us choose a sequence of functions $\varphi(z) \in \mathfrak{M}$ convergent in each point $z$ ($|z| < 1$) to $\varphi(z)$ and $\varphi_n(z)$ is represented by (3) where $S_n(t)$ have only a finished number of jumps. Each function $\varphi_n(C(\lambda))$ corresponds to a spectral function $\rho_n(t) \in \mathfrak{S}_0$. Owing to
the fact that $\mathcal{G} = \overline{\mathcal{G}_0}$, there is thus a sub-sequence $\rho_{n_k}(t)$ which converges in each point of continuity towards a function $\tilde{\rho}(t)$. As

$$
\frac{1 + \varphi(C(\lambda))C(t + i\tau)}{1 - \varphi(C(\lambda))C(t + i\tau)} = \int_{-\infty}^{+\infty} \frac{1}{t - \lambda} \, dp(t)
$$

and $\rho_{n_k}(t)$ and $\rho_{n_k}(C(\lambda))$ are connected by (12), while making tend $n_k \to \infty$ we will have:

$$
\rho_{n_k}(t) \to \tilde{\rho}(t) = \rho(t),
$$

thus $\rho(t) \in \mathcal{G}$. 

\[ \square \]

References


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