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ABSTRACT: This work presents corrections to “Blow-up directions at space infinity for solutions of semilinear heat equations” published in BSPM 23(2005), 9-28.

Key Words: nonlinear heat equation, blow-up at space infinity, blow-up direction.

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1. Introduction

In [2] we consider the equation

\[
\begin{align*}
\begin{cases}
  u_t &= \Delta u + f(u), & x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) &= u_0(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

and we had some results for the solution blowing up at space infinity. However, the assumptions of \( f \) and \( u_0 \) in [2, (1),(2), (5) and (6)] are too weak to achieve to the goal. Moreover, the statement of Theorem 3 (ii) is not precise. We shall correct these flaws.

We sent the revised version to the journal; however unfortunately the first version has been published. Moreover, the galley proof was not sent to the authors. We do not know the reason. It seems that there is a problem of e-mails.

First, we should change the condition of the nonlinear term \( f \) of (1) from

(A) “The nonlinear term \( f \) is assumed to be locally Lipschitz in \( \mathbb{R} \) with the property that

\[
\liminf_{s \to \infty} \frac{f(s)}{s^p} > 0 \quad \text{for some} \ p > 1, \ f' \geq 0.
\]

to a stronger condition:
(B) “The nonlinear term $f$ is assumed to be a nondecreasing function and locally Lipschitz in $\mathbb{R}$ with the property that

$$f(\delta b) \leq \delta^p f(b)$$

for all $b \geq b_0$ and for all $\delta \in (\delta_0, 1)$ with some $b_0 > 0$, some $\delta_0 \in (0, 1)$ and some $p > 1$.”

(The condition (B) is stronger than (A); it is easy to construct an example of $f$ a step-like function satisfies (A) but does not satisfy (B).) If this condition is fulfilled, $f$ satisfies (A) so that

$$\int_{-\infty}^{\infty} \frac{ds}{f(s)} < \infty,$$

(see Appendix), and a spatially constant solution of (1) blows up in finite time.

Secondly, we have to change the part of the assumptions of initial data $u_0$. It should be changed from

(C) “We assume that

$$\text{essinf}_{x \in B_m} (u_0(x) - M_m) \geq 0 \quad \text{for} \quad m = 1, 2, \ldots,$$

where

$$B_m = B_{r_m}(x_m)$$

with a sequence $\{r_m\}$ and a sequence of constants $M_m$ satisfying

$$\lim_{m \to \infty} r_m = \infty, \quad \lim_{m \to \infty} |M - M_m| = 0,$$

and $\{x_m\}_{m=1}^\infty$ is some sequence of vectors.”

to

(D) “We assume that

$$\text{essinf}_{x \in \tilde{B}_m} (u_0(x) - M_m(x - x_m)) \geq 0 \quad \text{for} \quad m = 1, 2, \ldots,$$

where

$$\tilde{B}_m = B_{r_m}(x_m)$$

with a sequence $\{r_m\}$, a sequence of vectors $\{x_m\}_{m=1}^\infty$ and a sequence of functions $\{M_m(x)\}$ satisfying

$$\lim_{m \to \infty} r_m = \infty, \quad M_m(x) \leq M_{m+1}(x) \quad \text{for} \quad m \geq 1$$

and

$$\lim_{m \to \infty} \inf_{x \in [1, r_m]} \frac{1}{|B_{r_m}|} \int_{B_{r_m}(0)} M_m(x) dx = M.$$
The condition (C) is not convenient to show Theorem 3.
Finally, we should correct Theorem 3 (ii) as follows;

(ii) If for any sequence $\{y_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \to \infty} y_m / |y_m| = \psi$, there exists a constant $c \in (1/(M + N), \infty)$ such that

$$\limsup_{m \to \infty} \inf_{s \in (1, c)} A_m(s) \leq M - \frac{1}{c},$$

then $\psi$ is not a blow-up direction.

We should correct the proof of Theorem 3. In fact we have to change the text from line 4 from below starting from “Finally, we must ...” in page 25 as follows.

Finally, we must show that the conditions of $\psi$ in (i) and (ii) cover all of $S^{n-1}$ exclusively. Let $\{s_m\}_{m=1}^{\infty}$ be a sequence satisfying $\lim_{m \to \infty} s_m = \infty$. We set $D = (1, \infty) \cap [1/(M + N), \infty)$ and the set of sequence

$$S(\psi) = \left\{ \{y_m\}_{m=1}^{\infty} \mid \lim_{m \to \infty} \frac{y_m}{|y_m|} = \psi, \lim_{m \to \infty} |y_m| = \infty \right\}.$$ 

Let $\Psi^\ast = \Psi^\ast(u_0)$ and $\Psi_\ast = \Psi_\ast(u_0)$ be the sets of directions of the form

$$\Psi^\ast = \left\{ \psi \in S^{n-1} \mid \exists \{y_m\}_{m=1}^{\infty} \in S(\psi), \limsup_{m \to \infty} \inf_{s \in (1, s_m)} A_m(s) = M \right\},$$

$$\Psi_\ast = \left\{ \psi \in S^{n-1} \mid \forall \{y_m\}_{m=1}^{\infty} \in S(\psi), \exists c \in D, \limsup_{m \to \infty} \inf_{s \in (1, c)} A_m(s) \leq M - \frac{1}{c} \right\}.$$ 

Here, $\Psi^\ast$ and $\Psi_\ast$ are the sets of all $\psi \in S^{n-1}$ satisfying, respectively, (i) and (ii) of Theorem 3.

Define two other sets $\Psi^\sharp = \Psi^\sharp(u_0)$ and $\Psi_\sharp = \Psi_\sharp(u_0)$ as follows:

$$\Psi^\sharp = \left\{ \psi \in S^{n-1} \mid \exists \{y_m\}_{m=1}^{\infty} \in S(\psi), \forall c \in D, \limsup_{m \to \infty} \inf_{s \in (1, c)} A_m(s) = M \right\},$$

$$\Psi_\sharp = \left\{ \psi \in S^{n-1} \mid \forall \{y_m\}_{m=1}^{\infty} \in S(\psi), \exists c \in D, \limsup_{m \to \infty} \inf_{s \in (1, c)} A_m(s) < M \right\}.$$ 

It is clear that $\Psi^\sharp = (\Psi_\sharp)^c$. We shall show that $\Psi^\ast = \Psi^\sharp$ and $\Psi_\ast = \Psi_\sharp$.

First we show $\Psi^\ast = \Psi^\sharp$. It is clear that $\Psi^\ast \subset \Psi^\sharp$. We shall show $\Psi^\ast \supset \Psi^\sharp$.

Take a sequence $\{c_l\}_{l=1}^{\infty} \subset \mathbb{R}$ such that $c_l < c_{l+1}$ for $l \geq 1$ and $\lim_{l \to \infty} c_l = \infty$. From the condition of $\Psi^\sharp$ we have

$$\lim_{m \to \infty} \inf_{s \in (1, c_l)} A_m(s) = M$$

for any $l \geq 1$. For $k \in \mathbb{N}$, take the subsequences $\{m_k\} \subset \{m\}$ and $\{l_k\} \subset \{l\}$ satisfying $m_k < m_{k+1}$, $l_k < l_{k+1}$, $\lim_{k \to \infty} c_l = \infty$ and $\lim_{k \to \infty} l_k = \infty$. We set $\tilde{c}_k = c_{l_k}$ to get

$$\lim_{k \to \infty} \inf_{s \in (1, \tilde{c}_k)} A_{m_k}(s) = M.$$
Thus we have $\Psi^* \supset \Psi^2$ and $\Psi^* = \Psi^2$.

Next, we show $\Psi_* = \Psi_z$. It is clear that $\Psi_* \subset \Psi_z$. We shall prove that $\Psi_* \supset \Psi_z$.

By the condition of $\Psi_z$ we have

$$\exists c \in D \text{ and } \exists \epsilon > 0, \lim_{m \to \infty} \inf_{s \in (1, c)} A_m(s) \leq M - \epsilon.$$ 

Take $c' = \max\{c, 1/\epsilon\}$. Then we have

$$\lim_{m \to \infty} \inf_{s \in (1, c')} A_m(s) \leq \frac{M - 1}{c'},$$ 

so we have $\Psi_* \supset \Psi_z$ and $\Psi_* = \Psi_z$.

Since $\Psi^* = \Psi_z$, $\Psi_* = \Psi_z$ and $\Psi^2 = (\Psi_z)^c$, we obtain $\Psi^* = (\Psi_*)^c$, and the proof is now complete.

We also mention that the proof of Lemma 3.7 of [2] is incomplete. A complete proof is given in [1, Lemma 4.2.1].

2. Appendix

We shall give a proof of that the condition (B) implies that the ODE $v_1 = f(v)$ blows up in finite time for sufficiently large initial data.

For the nonlinear term of the first equation of (1), we have one proposition.

**Proposition A.** Let $f$ be a continuous function. If $f$ satisfies (B), then (A) holds. In particular

$$\int_a^\infty \frac{ds}{f(s)} < \infty$$

for any $a \in \mathbb{R}$ satisfying $f(r) > 0$ for $r \geq a$.

**Proof:** We take $\delta_1 \in (\delta_0, 1)$ and $v_1 > b_0$. Since $\delta_1^{-\alpha_0} v_1 > b_0$ for $\alpha_0 \in [0, 1]$, we have

$$f(v_1) \leq \delta_1^{\alpha_0} f(\delta_1^{-\alpha_0} v_1)$$

by (2). Next, since $\delta_1^{-\alpha_0 - 1} v_1 > b_0$, we obtain

$$f(v_1) \leq \delta_1^{(\alpha_0 + 1)} f(\delta_1^{-\alpha_0 - 1} v_1)$$

by the same argument. By induction, we have

$$f(v_1) \leq \delta_1^{(\alpha_0 + N)} f(\delta_1^{-\alpha_0 - N} v_1)$$

for $\alpha_0$ and $n \in \mathbb{N}$. Then, we obtain (A). It is clear that

$$f(v_1) \leq \delta_1^{\alpha_0} f(\delta_1^{-\alpha} v_1)$$

for each $\alpha \geq 0$. Take $s = \delta_1^{-\alpha} v_1$ to get

$$f(v_1) \leq t^{-p} f(tv_1)$$
for \( s > v_1 > v_0 \). Thus we obtain
\[
\int_a^\infty \frac{ds}{f(s)} < \infty.
\]
The proof is now complete. 

3. List of typographical errors

There are other typographical errors we should correct.

1. P. 9, line 2 of ABSTRACT, “lim inf \( f(u)/u^p > 0 \)” \( \Rightarrow \) “\( f(\delta b) \leq \delta^p f(b) \) for all \( b \geq b_0 \) and all \( \delta \in (0, \delta_0) \) with some \( b_0 > 0 \), some \( \delta_0 \in (0, 1) \) and some \( p > 1 \).”

2. P. 13, line 1 from bottom and P.14, Line 2, “\( G_R(x, y, t) \)” \( \Rightarrow \) “\( \tilde{G}_R(x, y, t) \)”.

3. P. 14, line 7, line 19 and line 5 from bottom, “\( M_m \)” \( \Rightarrow \) “\( \tilde{M}_m(x - x_m) \)”.

4. P. 14, line 17, “\( X_m \leq X_{m+1} \)” \( \Rightarrow \) “\( X_m(x + x_m, t) \leq X_{m+1}(x + x_{m+1}, t) \)”.

From P. 14, line 19 to P. 15, line 12, all “\( G_m \)” \( \Rightarrow \) “\( \tilde{G}_m \)”.

5. P. 14, line 9 from bottom, “\( G_m(x, y, t) \) be the ... domain \( B_{R_m} \)” \( \Rightarrow \) “\( G_m = \tilde{G}_{R_m} \)”.

6. P. 14, line 4 from bottom and P. 15, line 2, after “\( \int_{\mathbb{R}^n} \)” , insert “\( G_m(x, y, t - s) \)”.

7. P. 15, line 3, remove “\{” between “...[\)” and “(\)”.

8. P. 15, line 6, “\( M_m \)” \( \Rightarrow \) “\( M_m(y - x_m) \)”.

9. P. 15, line 10 from bottom, insert “\( G_m(x, y, t - s) \)” between “\( f(t) \)” and “\( C \)”.

10. P. 19, line 3, “\( (M = 2, 3, \ldots) \)” \( \Rightarrow \) “\( (m = 2, 3, \ldots) \)”

11. P. 22, line 8, “\( \ldots \Delta w_m = \phi_m(f(u)) \ldots \)” \( \Rightarrow \) “\( \Delta w_m + \phi_m(f(u)) \ldots \)”.

12. P. 22, line 11, “\( g(x, s) \)” \( \Rightarrow \) “\( g_m(x, s) \)”.

13. P. 22, line 14, “\( e^{t\Delta} \)” \( \Rightarrow \) “\( e^{(t-s)\Delta} \)”.

14. P. 22, lines 18 and 19, “\( \|w'(\cdot, s)\|_{L^\infty(B_{\eta,m})} \)” \( \Rightarrow \) “\( \|w_m'(\cdot, s)\|_{L^\infty(B_{\eta,m})} \)”.

15. P. 22, line 4 from bottom, remove “\( \epsilon \)” between “\( C \)” and “\( \int_{\tau - \tau_0}^t \)”.

16. P. 23, line 12 from bottom, “\( \in (0, 1/2 - \epsilon^q - 1/(q - 1)) \)” \( \Rightarrow \) “\( \in [e^{q-1}/(q - 1) - 1/2, 0) \)”.

17. P. 23, line 11 from bottom, “\( m \in (2/(q - 1 - 2\epsilon^q - 1), 2/(q - 1 - 2\epsilon^q - 1) + 1) \)” \( \Rightarrow \) “\( m \in [(3 + 2\epsilon^q - q)/(q - 1 - 2\epsilon^q - 1), 2/(q - 1 - 2\epsilon^q - 1)] \)”.
18. P. 24, line 5, insert “for any $r \in (0, 1)$” after “then”.
19. P. 24, line 7, remove “$r \in (0, 1)$ and” between “with” and “$D$”.
20. P. 25, line 2, insert “in” between “as” and “proof”.
21. P. 25, line 12 from bottom and line 10 from bottom, “+” $\Rightarrow$ “−”.
22. P. 25, line 12 from bottom and line 10 from bottom, “$c_n$” $\Rightarrow$ “$c_m$”.

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