Some results about positive solutions of a nonlinear equation with a weighted Laplacian

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1. Introduction

We consider the problem of classification of bounded positive solutions to

\[(P)\quad -\nabla \cdot (A(|x|)\nabla u) = B(|x|)|u|^{q-2}u, \quad x \in \mathbb{R}^n.\]

Here \(q > 2\), and \(A, B\) are weight functions, i.e., a.e. positive measurable functions. Many authors have dealt with the non weighted case, i.e., with positive solutions to the equation

\[(E)\quad -\Delta u = |u|^{q-2}u, \quad x \in \mathbb{R}^n,\]

where \(q > 2\), see for instance [4].

In this case, when \(n > 2\), the critical number

\[2^* = \frac{2n}{n-2}\]

appears, and it is known that

\[\text{if } 1 < q < 2^*, \text{ all bounded solutions have a first positive zero,}
\]

\[\text{and if } q \geq 2^*, \text{ then the solutions are positive in } (0, \infty).\]

More recently, in 1993, the case of (E) with a weight in the right hand side, \(B(r) = \frac{1}{\gamma r^\gamma}, \gamma > 0\), that is the Matukuma equation, was studied by Ni-Yotsutani [10], Li-Ni [7, 8, 9], and Kawano-Yanagida-Yotsutani [5], where the problem

\[\text{is the problem of positive solutions to the equation}
\]

\[\text{where } q > 2, \text{ see for instance [4].}
\]

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\[\text{if } 1 < q < 2^*, \text{ all bounded solutions have a first positive zero,}
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\[\text{and if } q \geq 2^*, \text{ then the solutions are positive in } (0, \infty).\]
\[-(r^{n-1}u')' = \frac{r^{n-1}}{1+r^2}(u^+)q-1\]
\[u(0) = \alpha > 0\]  \hspace{1cm} (1.1)

is studied. The following result is due to Kawano-Yanagida-Yotsutani, [5], 1993:

**Theorem A.** Let $\gamma > 0$ and $n > 2$. Then

(i) If $2 < q \leq \max \{2, \frac{2(n-\gamma)}{n-2} \}$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.1) has a first positive zero in $(0, \infty)$.

(ii) If $q \geq \frac{2n}{n-2}$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.1) is positive in $(0, \infty)$ and $\lim_{r \to \infty} r^{n-2}u(r, \alpha) = \infty$.

(iii) If $\max \{p, \frac{2(n-\gamma)}{n-2} \} < q < \frac{2n}{n-2}$, then there exists a unique $\alpha^* > 0$ such that the solution $u(\cdot, \alpha)$ of (1.1) satisfies

- $u(r, \alpha) > 0$ for all $r > 0$ with $\lim_{r \to \infty} r^{n-2}u(r, \alpha) = \infty$ whenever $\alpha \in (0, \alpha^*)$.
- $u(r, \alpha^*) > 0$ for all $r > 0$ with $\lim_{r \to \infty} r^{n-2}u(r, \alpha^*) = \ell \in (0, \infty)$.
- $u(\cdot, \alpha)$ has a first zero for any $\alpha \in (\alpha^*, \infty)$.

Later, in 1995, Yanagida and Yotsutani [11] considered the case of a more general weight in the right hand side, and they studied the problem

\[-(r^{n-1}u')' = r^{n-1}K(r)(u^+)q-1\]
\[u(0) = \alpha > 0,\]  \hspace{1cm} (1.2)

for $K$ satisfying

(K1) \hspace{1cm} $K \in C^1(0, \infty), \ K > 0, \ rK(r) \in L^1(0,1),$

(K2) \hspace{1cm} $\frac{rK'(r)}{K(r)}$ decreasing and nonconstant in $(0, \infty)$.

They defined the critical numbers $-\infty \leq \ell < \sigma \leq \infty$

$$\sigma := \lim_{r \to 0} \frac{rK'(r)}{K(r)}, \ \ell := \lim_{r \to \infty} \frac{rK'(r)}{K(r)}, \ \sigma > -2, \ \sigma > \ell.$$  \hspace{1cm} (K1)

From (K1) $\sigma > -2$, and then they set

$$q_\sigma := \frac{2(n+\sigma)}{n-2}, \ q_\ell := \max \{2, \frac{2(n+\ell)}{n-2} \},$$

and proved the following:

**Theorem B.** Let $n > 2$ and assume that the weight $K$ satisfies (K1) and (K2). Then
(i) If $2 < q \le q_\ell$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.2) has a first positive zero in $(0, \infty)$.

(ii) If $q \ge q_\sigma$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.2) is positive in $(0, \infty)$ and $\lim_{r \to \infty} r^{n-2} u(r, \alpha) = \infty$.

(iii) If $q_\ell < q < q_\sigma$, then there exists a unique $\alpha^* > 0$ such that the solution $u(\cdot, \alpha)$ of (1.2) satisfies
\[ -u(r, \alpha) > 0 \text{ for all } r > 0 \text{ with } \lim_{r \to \infty} r^{n-2} u(r, \alpha) = \infty, \]
\[ -u(r, \alpha) > 0 \text{ for all } r > 0 \text{ with } \lim_{r \to \infty} r^{n-2} u(r, \alpha) = \ell \in (0, \infty) \text{ whenever } \alpha = \alpha^*. \]
\[ -u(\cdot, \alpha) \text{ has a first zero for any } \alpha \in (\alpha^*, \infty). \]

Clearly, the result in Theorem A is a particular case of that of Theorem B, since $K(r) = 1_{1+\gamma}^{-1}$ satisfies all the assumptions with $\sigma = 0$ and $\ell = -\gamma$.

We will deal here with the case $A = B$ in $(P)$ when the solutions are radially symmetric:
\[
(P_r) \quad \begin{cases} 
-(b(r)u')' = b(r)|u|^{q-2}u(r), & r \in (0, \infty), \\
\lim_{r \to 0} b(r)u'(r) = 0,
\end{cases}
\]
where $|x| = r$ and now the function $b(r) := r^{N-1}B(r)$ is a positive function satisfying some regularity and growth conditions. We will see in section 3 that under some extra assumption on the weight $K$ in (1.2), the problem considered in [11] is a particular case of ours.

Since we are interested only in positive solutions, we will study the initial value problem
\[
(IVP) \quad \begin{cases} 
-(b(r)u')' = b(r)(u^+)^{q-1}, & r \in (0, \infty), \\
u(0) = \alpha > 0, \lim_{r \to 0} b(r)u'(r) = 0.
\end{cases}
\]

Our note is organized as follows: in section 2 we will introduce some necessary conditions to deal with with our problem and we will state our main results which are a particular case of the work in [2]. Finally, in section 3 we compare our result with the one given in Theorem B.

2. Main results

We introduce next some necessary assumptions to deal with $(IVP)$. We note that if $u$ is a solution to our problem, then
\[ -b(r)u'(r) = \int_0^r b(s)(u^+)^{q-1}(s)ds > 0 \]
for all \( r > 0 \), and thus \( u'(r) < 0 \) for all \( r > 0 \). If for some positive \( R \) it happens that \( u(R) = 0, u(r) > 0 \) for \( r \in (0, R) \), then for all \( r \geq R \) and such that \( u(r) \leq 0 \), we have that

\[
|u'(r)| = (b(r))^{-1} \int_0^R b(s)(u^+)^{q-1}(s)ds
\]

and thus

\[
u(r) = -C \int_R^r (b(\tau))^{-1}d\tau < 0 \quad \text{for some positive constant } C.
\]

implying that \( u \) remains negative for all \( r \geq R \). If on the contrary it holds that \( u(r) > 0 \) for all \( r > 0 \), then

\[
|u'(r)| = (b(r))^{-1} \int_0^r b(s)(u^+)^{q-1}(s)ds,
\]

and thus, for \( r \geq s \) we have

\[
|u'(r)| \geq (b(r))^{-1} \int_0^s b(\tau)(u^+)^{q-1}(\tau)d\tau,
\]

implying that

\[
u(s) \geq \left( \int_0^s b(\tau)(u^+)^{q-1}(\tau)d\tau \right) \int_s^r (b(\tau))^{-1}d\tau,
\]

and we conclude that \( 1/b \in L^1(s, \infty) \) for all \( s > 0 \). Putting it in another way, if \( 1/b' \in L^1(1, \infty) \), then \( u \) must have a first positive zero. Therefore, keeping in mind that we are interested in the positive solutions to \((P_r)\), there is no loss of generality in assuming that \( 1/b \in L^1(s, \infty) \) for all \( s > 0 \).

Moreover, if \( u \) is any solution to our problem, then for \( r \geq s \) small enough it holds that

\[
b|u'(r)| - b|u'(s)| \geq \int_s^r b(\tau)d\tau,
\]

and thus

\[
b \in L^1(0, 1)
\]

is a necessary condition for the existence of solutions to \((IVP)\). Finally, it can be shown that

\[
\left( \int_0^r b(\tau)d\tau \right)(1/b) \in L^1(0, 1)
\]

is necessary and sufficient for the existence and uniqueness of solutions to \((IVP)\). Hence, our basic assumptions on the weight \( b \) will be:

\[
(H_1) \quad b \in C^1(\mathbb{R}^+, \mathbb{R}^+), (\mathbb{R}^+ = (0, \infty))
\]

\[
(H_2) \quad b \in L^1(0, 1), \quad 1/b \in L^1(1, \infty)
\]
Positive solutions

\[ \beta(r) := \int_0^r b(s) \, ds, \quad h(r) = \int_r^\infty (b(s))^{-1} \, ds, \]

\((H_3)\) \quad \((\beta/b) \in L^1(0, 1)\).

By a solution to \((IVP)\) we understand an absolutely continuous function \(u\) defined in the interval \([0, \infty)\) such that \(b(r)u'\) is also absolutely continuous in the open interval \((0, \infty)\) and satisfies the equation in \((IVP)\).

We will show that the behavior of function

\[ r \mapsto B_q(r) := \beta(r) h^{q/2}(r), \quad (2.1) \]

is crucial in the study of solutions to \((IVP)\). This function played a key role when studying the problem of existence of positive solutions to the corresponding Dirichlet problem associated to our equation, see \([1]\). The behavior at 0 of this function is closely related to the inclusion

\[ V_q^0(b) \hookrightarrow L^q(b) \]

of weighted Sobolev spaces. For a proper definition of these spaces we refer to Kufner-Opic \([6]\). Now also the behavior at \(\infty\) of this function will be crucial for our classification results. Let us define

\[ U := \{ s \geq 2 \mid \sup_{0 < r < 1} B_s(r) < \infty \}, \quad W := \{ s \geq 2 \mid \sup_{1 \leq r < \infty} B_s(r) < \infty \}, \]

and put

\[ \rho_0 = \sup U, \quad \rho_\infty = \inf W, \quad (2.2) \]

where we set \(\rho_\infty = \infty\) if \(W = \emptyset\). It can be proved that condition \((H_3)\) implies that \(2 \in U\) and thus \(U \neq \emptyset\). Observe that

\[ [2, \rho_0) \subseteq U, \quad (\rho_\infty, \infty) \subseteq W. \]

We will prove in section 2 that these critical numbers can be computed as

\[ \rho_0 = \max\left\{ 2, 2 \liminf_{r \to 0} \frac{\log(\beta(r))}{\log(h(r))} \right\}, \quad \rho_\infty = \max\left\{ 2, 2 \limsup_{r \to \infty} \frac{\log(\beta(r))}{\log(h(r))} \right\}. \]

We will denote the unique solution to \((IVP)\) by \(u(r, \alpha)\). As it is standard in the literature, we will say that

- \(u(r, \alpha)\) is a crossing solution if it has a zero in \((0, \infty)\).
- \(u(r, \alpha)\) is a slowly decaying solution if \(\lim_{r \to \infty} \frac{u(r)}{h(r)} = \infty\).
- \(u(r, \alpha)\) is a rapidly decaying solution if \(\lim_{r \to \infty} \frac{u(r)}{h(r)} = \ell \in (0, \infty)\).
In the case that $u$ is a crossing solution, we will denote its (unique) zero by $z(\alpha)$.

Our main results consist of a classification of the solutions according to the relative position of $q$ with respect to the critical values $\rho_0$ and $\rho_\infty$. In these results, the function
$$r \mapsto c(r) := 2b^2(r) \int_r^\infty (b(s))^{-1}ds$$
plays a fundamental role, the connection of this function with the critical values follows since
$$\lim \inf_{r \to 0} c(r) \leq \lim \inf_{r \to 0} 2|\log(\beta(r))| |\log(h(r))| = \rho_0,$$
and
$$\rho_\infty = 2\lim \sup_{r \to \infty} |\log(\beta(r))| |\log(h(r))| \leq \lim \sup_{r \to \infty} c(r).$$

Also, we note that in the non weighted case, that is, $b(r) = r^{n-1}, n > 2$, we have
$$\beta(r) = \frac{r}{n}, \quad h(r) = \frac{r^{2-n}}{n-2},$$
and thus
$$c(r) = \frac{2n}{n-2}.$$

Our first classification result generalizes the non weighted case:

**Theorem 2.1** Let the weight $b$ satisfy assumptions $(H_1), (H_2)$ and $(H_3)$. Let $q > 2$ be fixed and assume that $c(r) = 2b^2(r) \int_r^\infty (b(s))^{-1}ds \equiv \rho^*$. Then $h(0) = \infty$, $\rho^* > 2$ and

(i) If $q < \rho^*$, then $u(r, \alpha)$ a crossing solution for any $\alpha > 0$.

(ii) If $q = \rho^*$, then $u$ is the rapidly decaying solution given by
$$u(r, \alpha) = \left(\frac{C}{\alpha^{1-\frac{2}{n}} + h^{1-\frac{2}{n}}}\right)^{2/(\rho^*-2)},$$
where $C$ is a positive constant.

(iii) If $q > \rho^*$, then $u(r, \alpha)$ a slowly decaying solution for any $\alpha > 0$.

Finally, we generalize Theorem 2 in [11].

**Theorem 2.2** Let the weight $b$ satisfy assumptions $(H_1), (H_2)$ and $(H_3)$, and assume that they also satisfy

the function $r \mapsto c(r)$ is decreasing on $(0, \infty)$.

If $q \leq \rho_\infty$, then any solution of $(IVP)$ is crossing.

If $q \geq \rho_0$, then any solution of $(IVP)$ is slowly decaying.

If $\rho_\infty < q < \rho_0$, then there exists $\alpha^*$ such that
- \( u(\cdot, \alpha) \) is crossing for any \( \alpha > \alpha^* \).
- \( u(\cdot, \alpha^*) \) is rapidly decaying.
- \( u(\cdot, \alpha) \) is slowly decaying for any \( \alpha < \alpha^* \).

This result, as well as some very strong generalizations will appear in \([2]\).

### 3. Final remarks

In this section, we will compare our result in Theorem \([2,2]\) with Theorem B stated in the introduction. To this end, we will show that if in addition to \((K_1)\) and \((K_2)\), we assume that

\[
\frac{K_1}{2} \in L^1(0,1) \text{ and } K^{1/2} \not\in L^1(1,\infty), \tag{3.1}
\]

then the assumptions in Theorem \([2,2]\) are satisfied. Indeed, as in \([2]\), we make the change of variable

\[
r = r(t) := \int_0^t K^{1/2}(\tau)d\tau, \quad u(r) = v(t),
\]

and the problem

\[-(t^{n-1}v')' = t^{n-1}K(t)(v^+)^{q-1}, \quad t \in (0,\infty), \quad \left( t' = \frac{d}{dt} \right), \quad v(0) = \alpha > 0,
\]

is transformed into

\[-(b(r)u')' = b(r)(u^+)^{q-1}, \quad r \in (0,\infty), \quad \left( r' = \frac{d}{dr} \right), \quad u(0) = \alpha > 0,
\]

where

\[b(r) = t^{n-1}K^{1/2}(t).\]

By (3.1), \(r(0) = 0\) and \(r(\infty) = \infty\). Next, we will see that assumptions \((H_1)\), \((H_2)\), and \((H_3)\) are satisfied for this \(b\). Clearly, we only need to check that the first in \((H_2)\) and \((H_3)\) are satisfied. We begin by showing that \(b \in L^1(0,1)\). Indeed, by making the change of variable \(r = \int_0^t K^{1/2}(\tau)d\tau\), we find that

\[
\int_0^1 b(r)dr = \int_0^{t_1} t^{n-1}K(t)dt \\
\leq t_1^{n-2} \int_0^{t_1} tK(t)dt,
\]
where here and in the rest of this note, $t_1$ is defined by $1 = \int_0^{t_1} K^{1/2}(\tau) d\tau$, and thus $b \in L^1(0, 1)$. Also,

$$
\int_0^1 (b(r))^{-1} \left( \int_0^r b(\tau) d\tau \right) dr = \int_0^{t_1} t^{1-n} \left( \int_0^t s^{n-1} K(s) ds \right) dt \\
= \frac{t^{2-n}}{2-n} \int_0^t s^{n-1} K(s) ds \bigg|_{t=0}^{t_1} + \frac{1}{n-2} \int_0^{t_1} t K(t) dt \\
\leq \lim_{t \to 0} \frac{t^{2-n}}{n-2} \int_0^t s^{n-1} K(s) ds + \frac{1}{n-2} \int_0^{t_1} t K(t) dt \\
\leq \lim_{t \to 0} \frac{1}{n-2} \int_0^t s K(s) ds + \frac{1}{n-2} \int_0^{t_1} t K(t) dt \\
= \frac{1}{n-2} \int_0^{t_1} t K(t) dt,
$$

implying that $(H_3)$ holds.

Finally, we will see that under $(K_2)$, $c$ is decreasing, and thus our theorem applies: Indeed, it can be seen that in the variable $t$,

$$
c(r) = 2 \frac{b^2(r) \int_r^{\infty} (b(s))^{-1} ds}{\beta(r)} = 2 \frac{t^n K(t)}{n-2 \int_0^t s^{n-1} K(s) ds},
$$

and

$$
t \frac{c'(t)}{c(t)} + \frac{n-2}{2} c(t) = n + \frac{t K'(t)}{K(t)},
$$

hence, if $(K_2)$ holds, it must be that

$$
t \frac{c'(t)}{c(t)} + \frac{n-2}{2} c(t) \text{ is decreasing.}
$$

Hence, if $c'(t) > 0$ for $t \in (0, t_0)$, then $\frac{t c'(t)}{c(t)}$ must decrease in $(0, t_0)$. This, together with the fact that

$$
\lim_{t \to 0} \frac{t c'(t)}{c(t)} = 0,
$$

implies that $c'(t) < 0$ in $(0, t_0)$, a contradiction. Hence, there are points $t > 0$ in every interval $(0, t_0)$ where $c'(t) < 0$, implying that if $c$ is not always decreasing, it must have a minimum, which is not possible.

References


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