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Existence of Mild Solutions for Atangana-Baleanu Retarded Fractional Differential Equations with Nonlocal Conditions

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ABSTRACT: In the present paper, we investigate the existence of mild solutions for retarded semilinear fractional differential equations subject to nonlocal conditions in a Banach space. The results are obtained by applying the Schauder fixed point theorem and Grönwall's inequality, under suitable assumptions. Finally, an example is provided to illustrate and justify the theoretical results.

Keywords: Banach space, ABC-fractional derivative, fractional differential equations, Boundary value problem.

Contents

1. Introduction

We consider the following nonlocal initial value problem:

$$\begin{cases} {}^{ABC}D^{\varrho}\sigma(t) = A\sigma(t) + \chi(t, \sigma(t)), & t \in [0, b], \\ \sigma(t) = \phi(t) + \lambda(\sigma), & t \in [a, 0] \end{cases} \tag{1.1}$$

where the unknown function σ takes values in a separable Banach space X. The operator ${}^{ABC}D^{\varrho}$ represents the left Caputo AB-derivative of order ϱ, with $0 < \varrho < 1$, and A is the infinitesimal generator of a C_0-semigroup $T(t)$ on X. The functions χ and λ will be specified later and $\phi \in X$.

It is well known among researchers in applied mathematics that fractional calculus has experienced remarkable scientific development in recent years, despite having been introduced as early as 1695 by Newton and Leibniz. More recently, it has gained widespread recognition across various scientific disciplines as a more effective tool for modeling real-world phenomena than classical derivatives. The diversity of its applications is closely related to the wide range of existing definitions of fractional derivatives and integrals, including those of Riemann–Liouville, Hadamard, Grünwald–Letnikov, Caputo, Riesz–Caputo, Chen, Weyl, Erdélyi–Kober, among others. In 2015, Caputo and Fabrizio introduced a novel definition of nonlocal fractional derivatives with a non-singular kernel, formulated in the not necessarily Banach space $H^1(a, b)$. For $\varrho \in (0, 1)$, $\sigma \in H^1(a, b)$ with $b > a$, and a normalization function $B(\varrho)$ satisfying $B(0) = B(1) = 1$, the Caputo–Fabrizio fractional derivative of order ϱ is defined as follows:

$${}^{CF}D^{\varrho}_{a^+}\sigma(t) = \frac{B(\varrho)}{1-\varrho}\int_a^t \sigma'(s)\, e^{-\frac{\varrho}{1-\varrho}(t-s)}\, ds$$

Although this definition initially posed significant challenges for practical implementation, it was soon successfully applied in various fields, including thermal science, mechanical engineering, and groundwater studies. For further details, see [17,22,23] and the references therein. One year later, Atangana and Baleanu proposed a new class of nonlocal fractional derivatives characterized by a nonsingular kernel

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involving the Mittag–Leffler function. This approach generalizes the Caputo–Fabrizio derivative, which is constructed using an exponential-type kernel. The Atangana–Baleanu definition strengthened the connection between fractional calculus and the Mittag–Leffler function, highlighting their combined effectiveness in modeling complex phenomena. Additional discussions and applications can be found in [7,20,21,24].

The concept of nonlocal initial conditions was first introduced by Byszewski [11] in the context of first-order differential equations. Since then, considerable effort has been devoted to this topic, as nonlocal conditions often yield more realistic models than classical initial conditions. Numerous existence and solvability results have been established using fixed point methods in spaces such as $C([a,b],X)$; see, for example, [12,13,3].

The paper is structured as follows: Section 2 introduces the essential definitions and key results required for the subsequent analysis. Section 3 focuses on proving the existence of mild solutions to the differential equations under certain conditions. Lastly, Section 4 presents an example that demonstrates the theoretical findings.

2. Preliminary results

In this section, we introduce the fundamental preliminaries, notations, and auxiliary results that will be used throughout the paper, with particular emphasis on fractional integration and differentiation.

Let X be a Banach space endowed with the norm $\|.\|_X$. For $a<0<b$, we denote by $C([a,b];X)$ the Banach space of all continuous functions from $[a,b]$ into X, equipped with the supremum norm $\|\sigma\| = \sup_{t\in[a,b]}\|\sigma(t)\|_X$

Let $A:\mathcal{D}(A)\subset X \longrightarrow X$ be a linear unbounded operator with domain $\mathcal{D}(A)$. The resolvent set of A is denoted by $\rho(A)$, and for each $\mu\in\rho(A)$, the resolvent operator is defined by $R(\mu;A)=(\mu I-A)^{-1}$.

Definition 2.1 (See [4]) Let $p\in[1,+\infty)$, the Sobolev space $H^p(a,b)$ is defined by

$$H^p(a,b)=\{f\in L^2(a,b): D^\beta f\in L^2(a,b),\quad \text{for all } |\beta|\le p\}$$

Definition 2.2 (See [21]) Let $\sigma\in H^1(a,b)$, where $a\le b$, and let $0<\varrho<1$. The Caputo Atangana–Baleanu fractional derivative of order ϱ is defined as

$$\left({}^{ABC}D_a^\varrho\sigma\right)(t)=\frac{B(\varrho)}{1-\varrho}\int_a^t \sigma'(z)E_\varrho\left(-\varrho\frac{(t-z)^\varrho}{1-\varrho}\right)dz$$

where E_ϱ is the Mittag-Leffler function defined by $E_\varrho(t)=\Sigma_{n=0}^{\infty}\frac{t^n}{\Gamma(n\varrho+1)}$ and $B(\varrho)$ is a normalizing positive function that satisfies $B(0)=B(1)=1$.

The left Riemann Atangana-Baleanu fractional derivative of order ϱ is defined by

$$\left({}^{ABR}D_a^\varrho\sigma\right)(t)=\frac{B(\varrho)}{1-\varrho}\frac{d}{dt}\int_a^t \sigma(z)E_\varrho\left(-\varrho\frac{(t-z)^\varrho}{1-\varrho}\right)dz,$$

The associated fractional integral is expressed as

$${}^{AB}I_a^\varrho\sigma(t)=\frac{1-\varrho}{B(\varrho)}\sigma(t)+\frac{\varrho}{B(\varrho)\Gamma(\varrho)}\int_a^t(t-z)^{\varrho-1}\sigma(z)dz$$

Definition 2.3 (See [4]) The resolvent set of an operator A is defined by

$$\rho(A)=\{\mu\in\mathbb{C}\ :\ (\mu I-A):\mathcal{D}(A)\to X \text{ is bijective}\}.$$

For each $\mu\in\rho(A)$, the resolvent operator is given by

$$R(\mu,A):=(\mu I-A)^{-1},$$

which is a bounded linear operator on X.

Definition 2.4 (See [4]) An operator A is called sectorial if the following conditions hold:

1. A is a closed and linear operator.
2. There exist constants $M > 0$, $v \in \mathbb{R}$, and $u \in \left[\frac{\pi}{2}, \pi\right]$ such that

$$\Xi_{(u,v)} := \{\mu \in \mathbb{C} \ : \ \mu \neq v, \ |\arg(\mu - v)| < u\} \subset \rho(A).$$

3. The resolvent operator $R(\mu, A)$ satisfies the estimate

$$\|R(\mu, A)\| \leq \frac{M}{|\mu - v|}, \quad \forall \mu \in \Xi_{(u,v)}.$$

Theorem 2.1 (Gronwall's inequality [1]) *Let ς and σ be two integrable functions on $[a, b]$, and let Let $\eta \in C^1([a, b])$ be a continuous, non-negative, and non-decreasing function. Assume that $\varsigma \geq 0$ and $\sigma \geq 0$ for all $t \in [a, b]$.*

If

$$\varsigma(t) \leq \sigma(t) + \eta(t) \int_a^t (t - s)^{\varrho - 1} \varsigma(s) ds, \quad \forall t \in [a, b]$$

then

$$\varsigma(t) \leq \sigma(t) + \int_a^t \sum_{j=1}^{+\infty} \frac{[\Lambda(t)\Gamma(\varrho)]^j}{\Gamma(j\varrho)} (t - s)^{j\varrho - 1} \sigma(s) ds, \quad \forall t \in [a, b]$$

Theorem 2.2 (Schauder fixed point theorem [3]) *Let $\mathcal{V}$ be a nonempty, closed, convex, and bounded subset of a Banach space X, and let $\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be a continuous and compact mapping. Under these conditions, $\mathcal{T}$ must possess at least one fixed point in $\mathcal{V}$.*

3. Main result

This section is devoted to a detailed analysis aimed at establishing the existence of mild solutions for system (1.1). We begin by presenting the following remark, which plays a fundamental role in proving the main results.

Remark 3.1 *To formulate the mild solution of problem (1.1), we rely on an analytical procedure derived from the methods proposed in [2,16] for the study of a related Cauchy problem.*

$$\begin{cases} {}^{ABC}D^{\varrho}\sigma(t) = A\sigma(t) + g(t), \quad t \in [0, b], 0 < \varrho < 1 \\ \sigma(0) = \sigma_0 \in X. \end{cases}$$

The above problem admits a mild solution given by

$$\sigma(t) = \Upsilon T_{\varrho}(t)\sigma_0 + \frac{\Upsilon\varphi(1 - \varrho)}{B(\varrho)\Gamma(\varrho)} \int_0^t (t - s)^{\varrho - 1} g(s) ds + \frac{\varrho\Upsilon^2}{B(\varrho)} \int_0^t S_{\varrho}(t - s) g(s) ds$$

where Υ and φ are bounded linear operators defined by

$$\Upsilon = \zeta(\zeta I - A)^{-1}, \quad \varphi = -\gamma A(\zeta I - A)^{-1}$$

with $\zeta = \frac{B(\varrho)}{1-\varrho}, \gamma = \frac{\varrho}{1-\varrho}$, Moreover, the operator families $T_{\varrho}(t)$ and $S_{\varrho}(t)$ are defined by

$$T_{\varrho}(t) = E_{\varrho}\left(-\varphi t^{\varrho}\right) = \frac{1}{2\pi i} \int_{\Gamma} e^{t\tau} \tau^{(\varrho - 1)} \left(\tau^{\varrho} I - \varphi\right)^{-1} d\tau$$

$$S_{\varrho}(t) = t^{\varrho - 1} E_{\varrho,\varrho}\left(-\varphi t^{\varrho}\right) = \frac{1}{2\pi i} \int_{\Gamma} e^{t\tau} \left(\tau^{\varrho} I - \varphi\right)^{-1} d\tau$$

where Γ is a suitable contour contained in $\Sigma_{(\mu,v)}$ and $g \in C([a, b], X)$. For further details, we refer the reader to [14]. where Γ is a suitable contour lying on $\Sigma_{(\mu,v)}$ and $g \in C([a, b], X)$. See [14]

Based on the preceding discussion, we introduce the following definition.

Definition 3.1 Let $\chi \in C([0,b] \times X, X)$. A function $\sigma : [a,b] \longrightarrow X$ is called a mild solution of problem (1.1) if

$$\sigma(t) = \begin{cases} \Upsilon T_\varrho(t)\left(\phi(0) - \lambda(\sigma)\right) + \frac{\Upsilon\varphi(1-\varrho)}{B(\varrho)\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1}\chi(s,\sigma(s))ds \\ +\frac{\varrho\Upsilon^2}{B(\varrho)} \int_0^t S_\varrho(t-s)\chi(s,\sigma(s))ds, \qquad t \in (0,b] \\ \phi(t) + \lambda(\sigma), \qquad t \in [a,0] \end{cases}$$

where $\Upsilon = \zeta(\zeta I - A)^{-1}$, $\varphi = -\gamma A(\zeta I - A)^{-1}$, $\zeta = \frac{B(\varrho)}{1-\varrho}$ and $\gamma = \frac{\varrho}{1-\varrho}$.

Lemma 3.1 (See [2]) *If $A \in A^\varrho(\mu_0, v_0)$ then $\|T_\varrho(t)\| \leq Me^{tv}$ and $\|S_\varrho(t)\| \leq Ce^{tv}\left(1 + t^{\varrho-1}\right)$, for all $t > 0, v > v_0$.*

According to the above lemma, if we set

$$\eta_1 = \sup_{t\in(0,b]} \|T_\varrho(t)\| \text{ and } \eta_2 = \sup_{t\in(0,b]} Ce^{tv}\left(1 + t^{\varrho-1}\right)$$

. then it follows that

$$\|T_\varrho(t)\| \leq \eta_1 \text{ and } \|S_\varrho(t)\| \leq t^{\varrho-1}\eta_2$$

For further details, see [2].

The following assumptions are used to establish the existence results.
(H_1) χ satisfies Carathedory condition i.e. $\chi(\cdot,\sigma) : [0,b] \longrightarrow X$ is Lebesgue measurable and $\chi(t,\cdot) : X \longrightarrow X$ is continuous.
(H_2) For every $\beta > 0$ there exists a function $h \in L^\infty([0,b]; \mathbb{R}^+)$ satisfying

$$\sup_{\|\sigma(t)\|_X \leq \beta} \|\chi(t,\sigma(t))\|_X \leq h(t) \quad \text{for a.e. } t \in [0,b],$$

(H_3) $T_\varrho(t)$ and $S_\varrho(t)$ are compact operators for all $t \in [0,b]$.
(H_4) The function $\lambda : X \longrightarrow X$ is continuous. For any $\beta > 0$, there exist $k_\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ sutch that

$$\|\lambda(\sigma)\|_X \leq k_\beta(\|\sigma\|_X)$$

and

$$\lim_{\beta\longrightarrow+\infty} \frac{k_\beta(\|\sigma\|_X)}{\beta} := \delta < +\infty$$

(H_5) There exist $N_0 > 0$, such that $\forall\sigma_1, \sigma_2 \in X$,

$$\|\chi(t,\sigma_1) - \chi(t,\sigma_2)\|_X \leq N_0\|\sigma_1 - \sigma_2\|_X$$

Theorem 3.1 *Suppose that assumptions $(H_1) - (H_4)$ hold and that*

$$(1 + \eta_1 \|\Upsilon\|_X)\delta \leq 1 \tag{3.1}$$

Then the fractional semilinear differential equation (1.1) has at least one mild solution in X. Moreover, if (H_5) holds, this mild solution is unique.

Proof:

Let $\beta > 0$ be given, and define the closed, bounded subset of $C([a,b], X)$ by $\mathcal{B}_\beta = \{\sigma \in C([a,b], X) : \|\sigma\|_X \leq \beta\}$

Next, we consider the operator $\Lambda : C([a,b]; X) \longrightarrow C([a,b]; X)$ defined by $\Lambda = \Lambda_1 + \Lambda_2$, where

$$\Lambda_1\sigma(t) = \begin{cases} 0, & t \in (0,b] \\ \phi(t) + \lambda(\sigma), & t \in [a,0] \end{cases}$$

and

$$\Lambda_2\sigma(t)=\begin{cases}\Upsilon T_\varrho(t)\left(\phi(0)-\lambda(\sigma)\right)+\frac{\Upsilon\varphi(1-\varrho)}{B(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}\chi(s,\sigma(s))ds\\ +\frac{\varrho\Upsilon^2}{B(\varrho)}\int_0^t S_\varrho(t-s)\chi(s,\sigma(s))ds, & t\in(0,b]\\ 0, & t\in[a,0]\end{cases}$$

The proof will be given in the following steps.

Step 1. We show that $\Lambda\left(B_\beta\right)\subseteq B_\beta$.

Assume that this is not the case. Then, for every $\beta>0$, there exist $\sigma\in B_\beta$, and $t\in(0,T]$ such that $\|(\Lambda\sigma)(t)\|_X>\beta$.

Since

$$\|\left(\Lambda_1\sigma\right)(t)\|_X\leq\|\phi\|_X+k_\beta\left(\|\sigma\|_X\right),$$

and

$$\begin{aligned}\|\left(\Lambda_2\sigma\right)(t)\|_X &\leq \|\Upsilon T_\varrho(t)(\phi(0)+\lambda(\sigma))\|_X+\frac{\varrho\|\Upsilon^2\|_X}{B(\varrho)}\int_0^t\|S_\varrho(t-s)\chi\left(s,\sigma(s)\right)\|_X\,ds\\ &+\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{B(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}\|\chi\left(s,\sigma(s)\right)\|_X\,ds\\ &\leq\eta_1\|\Upsilon\|_X\left(\|\phi(0)\|_X+k_\beta\left(\|\sigma\|_X\right)\right)+\frac{\eta_2\varrho\|\Upsilon\|_X^2}{B(\varrho)}\int_0^t(t-s)^{\varrho-1}\|\chi\left(s,\sigma(s)\right)\|_X\,ds\\ &+\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{B(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}\|\chi\left(s,\sigma(s)\right)\|_X\,ds\\ &\leq\eta_1\|\Upsilon\|_X\left(\|\phi\|_X+k_\beta\left(\|\sigma\|_X\right)\right)+\left(\eta_2\|\Upsilon\|_X^2+\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{\Gamma(\varrho+1)}\right)\frac{\|h\|_{L^\infty}b^\varrho}{B(\varrho)}\end{aligned}$$

Then

$$\begin{aligned}\beta<\|\left(\Lambda\sigma\right)(t)\|_X &\leq\|\left(\Lambda_1\sigma\right)(t)\|_X+\|\left(\Lambda_2\sigma\right)(t)\|_X\\ &\leq\|\phi(t)\|_X+k_\beta\left(\|\sigma\|_X\right)+\eta_1\|\Upsilon\|_X\left(\|\phi\|_X+k_\beta\left(\|\sigma\|_X\right)\right)\\ &+\left(\|\Upsilon\|_X^2+\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{\Gamma(\varrho+1)}\right)\frac{\|h\|_{L^\infty}b^\varrho}{B(\varrho)}\\ &\leq\left(\eta_1\left\|\Upsilon\right\|_X+1\right)\left(\|\phi\|_X+k_\beta\left(\|\sigma\|_X\right)\right)+\left(\|\Upsilon\|_X^2+\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{\Gamma(\varrho+1)}\right)\frac{\|h\|_{L^\infty}b^\varrho}{B(\varrho)}\end{aligned}$$

After dividing both sides by β and passing to the sup limit as $\beta\longrightarrow+\infty$, we arrive at

$$1<(1+\eta_1\left\|\Upsilon\right\|_X)\delta$$

which is a contradiction with (3.1). Consequently, $\Lambda(B_\beta)\subset B_\beta$

Step 2: $\Lambda(B_\beta)$ is uniformly bounded.

This is clear since $\Lambda(B_\beta)\subset B_\beta$ is bounded.

Step 3: Λ is continuous.

Let $\left(\sigma_n\right)_{n\in\mathbb{N}}$ be a sequence of B_β such that $\sigma_n\to\sigma$ as $n\to\infty$ in B_β. We aim to prove that $\Lambda\sigma_n\to\Lambda\sigma$ as $n\to\infty$ in B_β.

Its clear that $\|\Lambda_1\sigma_n(t)-\Lambda_1\sigma(t)\|_X\longrightarrow$ as $n\longrightarrow+\infty$. On the other hand, from Lemma 3.1, it follows that:

$$\begin{aligned}\left\|\Lambda_2\sigma_n(t)-\Lambda_2\sigma(t)\right\|_X \le & \left\|\Upsilon T_\varrho(t)(\lambda(\sigma_n)-\lambda(\sigma))+\frac{\Upsilon\varphi(1-\varrho)}{\Gamma(\varrho)B(\varrho)}\int_0^t (t-s)^{\varrho-1}\chi\left(s,\sigma_n(s)\right)-\chi\left(s,\sigma(s)\right)ds\right. \\ & \left.+\frac{\varrho\Upsilon^2}{B(\varrho)}\int_0^t S_\varrho(t-s)\left[\chi\left(s,\sigma_n(s)\right)-\chi\left(s,\sigma(s)\right)\right]ds\right\|_X \\ \le & \left\|\Upsilon\right\|_X \|T_\varrho(t)\|_X \left\|(\lambda(\sigma_n)-\lambda(\sigma))\right\|_X \\ & +\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{\Gamma(\varrho)B(\varrho)}\int_0^t (t-s)^{\varrho-1}\left\|\chi\left(s,\sigma_n(s)\right)-\chi\left(s,\sigma(s)\right)\right\|_X ds \\ & +\frac{\varrho\left\|\Upsilon^2\right\|_X}{B(\varrho)}\int_0^t \left\|S_\varrho(t-s)\right\|_X\left\|\chi\left(s,\sigma_n(s)\right)-\chi\left(s,\sigma(s)\right)\right\|_X ds \\ \le & \eta_1\left\|\Upsilon\right\|_X\left\|(\lambda(\sigma_n)-\lambda(\sigma))\right\|_X \\ & +\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{\Gamma(\varrho)B(\varrho)}\int_0^t (t-s)^{\varrho-1}\left\|\chi\left(s,\sigma_n(s)\right)-\chi\left(s,\sigma(s)\right)\right\|_X ds \\ & +\frac{\varrho\left\|\Upsilon\right\|_X^2}{B(\varrho)}\eta_2\int_0^t (t-s)^{\varrho-1}\left\|\chi\left(s,\sigma_n(s)\right)-\chi\left(s,\sigma(s)\right)\right\|_X ds\end{aligned}$$

taking the continuity of the functions χ and λ into consideration and Lebesgue dominated convergence theorem, we get:

$$\left\|\Lambda_2\sigma_n(t)-\Lambda_2\sigma(t)\right\|_X \to 0 \text{ as } n\to\infty.$$

Consequently, the operator $\Lambda_2 : B_\beta \to X$ is continuous.
Step 4. Λ is equicontinuos

$$\begin{aligned}\left\|\Lambda_2\sigma(t_1)-\Lambda_2\sigma(t_2)\right\|_X \le & \left\|\Upsilon\right\|_X\left\|T_\varrho(t_1)-T_\varrho(t_2)\right\|_X\left(\|\phi(0)\|_X+\|\lambda(\sigma)\|_X\right) \\ & +\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{B(\varrho)\Gamma(\varrho)}\left\|\int_0^{t_1}(t_1-s)^{\varrho-1}\chi(s,\sigma(s))ds-\int_0^{t_2}(t_2-s)^{\varrho-1}\chi(s,\sigma(s))ds\right\|_X \\ & +\frac{\varrho\left\|\Upsilon^2\right\|_X}{B(\varrho)}\left\|\int_0^{t_1}S_\varrho(t_1-s)\chi(s,\sigma(s))ds-\int_0^{t_2}S_\varrho(t_2-s)\chi(s,\sigma(s))ds\right\|_X \\ \le & W_1+\frac{\|\Upsilon\|_X\|\varphi\|_X(1-\varrho)}{B(\varrho)\Gamma(\varrho)}W_2+\frac{\varrho\|\Upsilon\|^2}{B(\varrho)}W_3\end{aligned}$$

By using the hypothesis (H_3), we get

$$W_1=\left\|\Upsilon\right\|_X\left\|T_\varrho(t_1)-T_\varrho(t_2)\right\|_X\left(|\phi(0)|+\|\lambda(\sigma)\|_X\right)\longrightarrow 0 \textbf{ as } t_1\longrightarrow t_2$$

In addition

$$\begin{aligned}W_2 \le & \left\|\int_0^{t_2}(t_1-s)^{\varrho-1}\chi(s,\sigma(s))ds+\int_{t_2}^{t_1}(t_1-s)^{\varrho-1}\chi(s,\sigma(s))ds-\int_0^{t_2}(t_2-s)^{\varrho-1}\chi(s,\sigma(s))ds\right\|_X \\ \le & \int_{t_2}^{t_1}(t_1-s)^{\varrho-1}\|\chi(s,\sigma(s))\|_X ds+\int_0^{t_2}\left|(t_1-s)^{\varrho-1}-(t_2-s)^{\varrho-1}\right|\|\chi(s,\sigma(s))\|_X ds \\ \le & \|h\|_{L^\infty}\left(\int_{t_2}^{t_1}(t_1-s)^{\varrho-1}ds+\int_0^{t_2}\left|(t_1-s)^{\varrho-1}-(t_2-s)^{\varrho-1}\right|ds\right) \\ \le & \frac{\|h\|_{L^\infty}}{\varrho}\left((t_1-t_2)^\varrho+t_1^\varrho-t_2^\varrho+(t_1-t_2)^\varrho\right)\end{aligned}$$

$\le \frac{\|h\|_{L^\infty}}{\varrho}\left(t_1^\varrho-t_2^\varrho+2(t_1-t_2)^\varrho\right)$
Hence, $W_2 \longrightarrow 0$ **as** $t_1\longrightarrow t_2$

$$
\begin{aligned}
W_3 &= \left\| \int_0^{t_1} S_\varrho\left(t_1-s\right) \chi(s,\sigma(s))ds - \int_0^{t_2} S_\varrho\left(t_2-s\right) \chi(s,\sigma(s))ds \right\|_X \\
&\leq \left| \int_0^{t_1} \left\|S_\varrho\left(t_1-s\right)\right\|_X \|\chi(s,\sigma(s))\|_X ds - \int_0^{t_2} \left\|S_\varrho\left(t_1-s\right)\right\|_X \|\chi(s,\sigma(s))\|_X ds \right| \\
&\leq \eta_2 \left| \int_0^{t_2} \left(t_1-s\right)^{\varrho-1} \|\chi(s,\sigma(s))\|_X ds + \int_{t_2}^{t_1} \left(t_2-s\right)^{\varrho-1} \|\chi(s,\sigma(s))\|_X ds \right. \\
&\left. - \int_0^{t_2} \left(t_2-s\right)^{\varrho-1} \|\chi(s,\sigma(s))\|_X ds \right| \\
&\leq \eta_2 \int_{t_2}^{t_1} \left(t_1-s\right)^{\varrho-1} \|\chi(s,\sigma(s))\|_X ds + \eta_2 \left| \int_0^{t_2} \left(\left(t_1-s\right)^{\varrho-1} - \left(t_2-s\right)^{\varrho-1}\right) \|\chi(s,\sigma(s))\|_X ds \right| \\
&\leq \eta_2 \|h\|_{L^\infty} \left(\int_{t_2}^{t_1} \left(t_1-s\right)^{\varrho-1} ds + \int_0^{t_2} \left| \left(t_1-s\right)^{\varrho-1} - \left(t_2-s\right)^{\varrho-1} \right| ds \right) \\
&\leq \eta_2 \frac{\|h\|_{L^\infty}}{\varrho} \left(t_1^\varrho - t_2^\varrho + 2(t_1-t_2)^\varrho\right)
\end{aligned}
$$

Therefore, $\|\Lambda_2\sigma(t_1) - \Lambda_2\sigma(t_2)\|_X \longrightarrow 0$ as $t_1 \longrightarrow t_2$ this implies that $\Lambda(B_\beta)$ is equicontinuous. Therefore, by Steps 1–4, along with the Arzelà–Ascoli theorem, we conclude that the operator Λ is continuous and compact. Consequently, by Schauder's fixed-point theorem, Λ has a fixed point σ, which is a solution to the equation (1.1).

Step 5.To demonstrate the uniqueness of the fixed point of Λ, assume that σ_1 and σ_2 are two fixed points of Λ:

$$
\begin{aligned}
\|\sigma_1(t) - \sigma_2(t)\|_X &= \left\| \frac{\Upsilon\varphi(1-\varrho)}{B(\varrho)\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} \chi\left(s,\sigma_n(s)\right) - \chi\left(s,\sigma(s)\right) ds \right. \\
&\left. + \frac{\varrho\Upsilon^2}{B(\varrho)} \int_0^t S_\varrho(t-s) \left[\chi\left(s,\sigma_1(s)\right) - \chi\left(s,\sigma_2(s)\right)\right] ds \right\|_X \\
&\leq \frac{\|\Upsilon\|_X \|\varphi\|_X (1-\varrho)}{B(\varrho)\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} \left\|\chi\left(s,\sigma_1(s)\right) - \chi\left(s,\sigma_2(s)\right)\right\|_X ds \\
&+ \frac{\varrho\left\|\Upsilon^2\right\|}{B(\varrho)} \int_0^t \|S_\varrho(t-s)\|_X \left\|\chi\left(s,\sigma_1(s)\right) - \chi\left(s,\sigma_2(s)\right)\right\|_X ds \\
&\leq \frac{\|\Upsilon\|_X \|\varphi\|_X (1-\varrho)}{B(\varrho)\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} \left\|\chi\left(s,\sigma_1(s)\right) - \chi\left(s,\sigma_2(s)\right)\right\|_X ds \\
&+ \frac{\varrho\left\|\Upsilon^2\right\|}{B(\varrho)} \eta_2 \int_0^t (t-s)^{\varrho-1} \left\|\chi\left(s,\sigma_1(s)\right) - \chi\left(s,\sigma_2(s)\right)\right\|_X ds \\
&\leq \left(\frac{\|\Upsilon\|_X \|\varphi\|_X (1-\varrho)}{B(\varrho)\Gamma(\varrho)} + \frac{\varrho\left\|\Upsilon^2\right\|}{B(\varrho)} \eta_2 \right) \int_0^t (t-s)^{\varrho-1} \left\|\chi\left(s,\sigma_1(s)\right) - \chi\left(s,\sigma_2(s)\right)\right\|_X ds
\end{aligned}
$$

Then

$$
\|\sigma_1(t) - \sigma_2(t)\|_X \leq \left(\frac{\|\Upsilon\|_X \|\varphi\|_X (1-\varrho)}{B(\varrho)\Gamma(\varrho)} + \frac{\varrho\left\|\Upsilon^2\right\|}{B(\varrho)} \eta_2 \right) N_0 \int_0^t (t-s)^{\varrho-1} \|\sigma_1(s) - \sigma_2(s)\|_X ds
$$

By using Grönwall's inequality , we obtain that

$$
\|\sigma_1(t) - \sigma_2(t)\|_X = 0, \quad \forall t \in [a,b]
$$

which implies that $\sigma_1 = \sigma_2$.

Thus, the operator Λ possesses a unique fixed point, which is also the unique mild solution to the system (1.1). □

4. An application

Take the following semilinear differential equation of fractional order:

$$\begin{cases} {}^{ABC}\partial_t^{\frac{1}{2}}\sigma(t,x) = \frac{\partial}{\partial x^2}\sigma(t,x) + \frac{e^{-t}}{1+e^t}\sigma(t,x), t \in [0,1], x \in [0,\pi] \\ \sigma(t,0) = \sigma(t,\pi) = 0 \;\; t \in [0,1] \\ \sigma(t) = \phi(t) + \Sigma_{i=1}^{3}\frac{1}{4^i}z(\frac{1}{i},x), \;\; t \in [-1,0], x \in [0,\pi] \end{cases} \tag{4.1}$$

Consider the Banach space $X = L^2[0,\pi]$ and define the linear operator $A : \mathcal{D}(A) \subset X \to X$ by $A\sigma = \sigma''$, for $\sigma \in \mathcal{D}(A) = \{\sigma \in X : \; \sigma, \sigma' \text{ are absolutely continuous}, \; \sigma(0) = \sigma(\pi) = 0\}$.
For $\sigma \in \mathcal{D}(A)$, we have

$$A\sigma = \sum_{n=1}^{\infty} n^2 \langle \sigma, \sigma_n \rangle \sigma_n.$$

where $(\sigma_n)_{n\in\mathbb{N}}$ is the orthonormal set of eigenvectors given by

$$\sigma_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), \qquad n \in \mathbb{N}.$$

Moreover, A generates a compact C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X given by

$$T(t)\sigma = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \sigma, \sigma_n \rangle \sigma_n, \qquad \sigma \in X,$$

which is uniformly bounded by 1. hence the operator $R(\mu, A) = (\mu I - A)^{-1}$ is compact for each $\mu \in \rho(A)$ i.e. $A \in A^{\varrho}(\mu_0, \nu_0)$. Also, by subordination principle of solution operator, we have $\|T_{\varrho}(t)\|_{L^2} \leq 1$ for each $t \in [0,1]$.

Let $\sigma(t) = \sigma(t,\cdot) \in X$. Then the given fractional partial differential equation can be rewritten as the abstract form (1.1). with

the functions χ, λ are defined by $\chi(t,\sigma) = \frac{e^{-t}}{1+e^t}\sigma(t,\cdot)$, and $\lambda(\sigma) = \sum_{i=1}^{3}\frac{1}{4^i}\sigma(\frac{1}{i},\cdot)$ and it satisfies all the assumptions $(H_1) - (H_5)$, so the given system (4.1) has unique mild solution.

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